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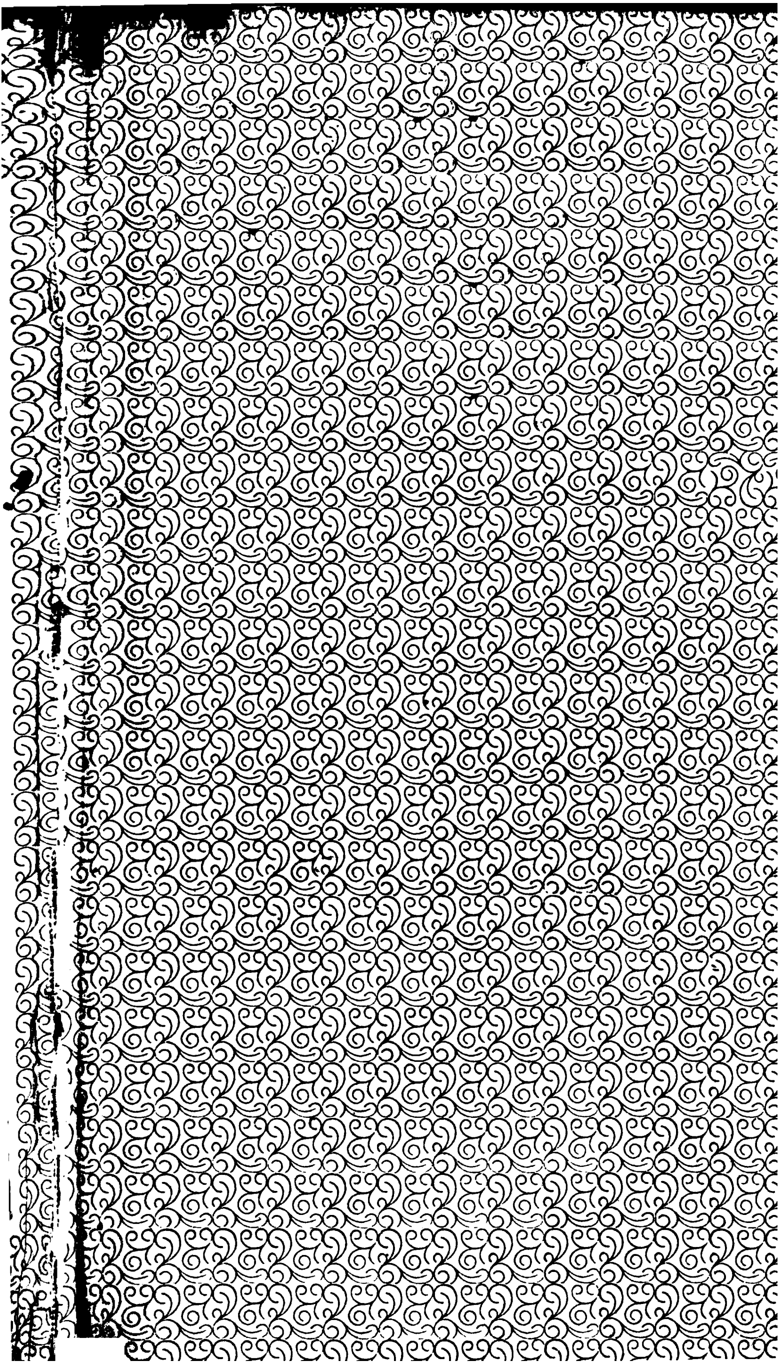

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THE

# ANALYSIS AND SOLUTION

OF

# CUBIC AND BIQUADRATIC EQUATIONS.

PRINTED BY C. ADLARD, BARTHOLOMEW CLOSE.

# THE

# ANALYSIS AND SOLUTION

# OF

# CUBIC AND BIQUADRATIC EQUATIONS;

FORMING A SEQUEL TO

THE ELEMENTS OF ALGEBRA,

AND AN INTRODUCTION TO THE

THEORY AND SOLUTION OF EQUATIONS OF THE HIGHER ORDERS.

ohn adford

BY J. R. YOUNG,

PROFESSOR OF MATHEMATICS IN BELFAST COLLEGE.

LONDON:
SOUTER AND LAW, 131, FLEET STREET.
1842.

# PREFACE.

Considerable advances have of late been made in that department of analysis to which the theory and solution of Numerical Equations belongs; although, as is often the case with extensive innovations upon long-established practices, these improvements have met with general acceptance by but very slow degrees.

In a work on Equations, published in 1835, I endeavoured to embody the principal of the modern discoveries in this department of enquiry, in a form accessible to the algebraic student; and from the various public testimonies which have appeared in commendation of that endeavour, I am encouraged to hope that the book has in some degree contributed to awaken that spirit of examination into the pretensions and capabilities of the modern methods, which is now in pretty active exercise among British mathematicians; and from which it may be reasonably expected that these methods will be eventually perfected into the forms best suited to the actual demands of practical science.

On account of the variety of topics discussed in the publication just adverted to, all of them closely connected with the general doctrine of equations, it was found that brevity and compression were essential to the design of comprehending the enquiry within the compass of a single volume of moderate size and price.

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The important theorems of Sturm and Horner, more especially the former, were thus perhaps too hastily disposed of; and many interesting details and developments omitted which might have placed the practical value of these theorems more conspicuously and more favorably before the reader.

But at that time the theorem of Sturm was new: and I must acknowledge that I had but too imperfectly examined into its capabilities of practical improvement to justify my venturing upon any addition to what the author himself had delivered.* I have since more deliberately considered the subject, and have offered some suggestions for facilitating the numerical process which the practice of the theorem requires. These suggestions will be found in the *Mathematical Dissertations*, recently published; and others of a different kind, but aiming at the same object, are given in the volume now submitted to the public.

---

* Sturm's discovery was not known to me till my book on Equations had been some time at press, and was in great part printed. I believe the fame of it had not reached England at that time, as the publication of my own book seems to have furnished the first announcement of it in the English language. I may further mention, in order to escape the charge of hasty negligence, to which the admission in the text may expose me, that the first intimation I received of the theorem was in May, 1835, through the last sheet of an edition of the *Complément d'Algèbre,* then at press, and given by Lacroix to a friend of mine (Mr. Spiller) at that time at Paris. Lacroix spoke somewhat disparagingly of the discovery; but I thought it of sufficient importance to justify the destruction of many pages of my MS. and the suspension of the work till the volume containing Sturm's paper should be published. This volume —a ponderous quarto of upwards of 1000 pages—was forwarded to me from Paris, by post, in July; and my own treatise was published in the following August. I trust it will not be inferred, from this explanatory note, that I consider any merit to be due to the mere importing of a discovery from one country to another: yet it may be well to have shown that the importation is not always unattended with inconvenience, anxiety, and expense.

In this volume I have endeavoured to unfold, with all the amplitude that can be desired, as much of the theory and solution of Numerical Equations as can reasonably be expected to enter into an ordinary course of mathematical education. I wish it to be regarded as a sequel to elementary Algebra; and as a comprehensive introduction to the more advanced volume on the *Theory of Equations of the higher orders*, a new and enlarged edition of which will be shortly published.

As the discoveries of STURM and HORNER are destined to become permanently incorporated among the doctrines of analysis, I have spared no pains to present them to the young student with all due copiousness of illustration and practical detail. The theoretical exposition of them is different from that delivered by the authors themselves, being more simple and elementary; whilst a larger number of practical applications of them have been given than any other work affords.

These numerical illustrations have involved me in considerable labour, as a glance at the closing sheets of the work will render sufficiently apparent.

This labour might however have been greatly reduced by the exercise of a little management, in framing the examples in favorable adaptation to the methods employed. I have not adopted any manœuvres of this kind; but have, on the contrary, been anxious to submit the new methods to the severest tests, by applying them to the most unfavorable and difficult examples I could find.

The plan that I have recommended at page 111 for conducting the operation for analysing an equation by the method of Sturm, is different from, and somewhat longer than that proposed in the *Dissertations*. It is not to be preferred to this latter process on account of superior brevity—which it has not; but merely because its more elementary simplicity gives it a better claim to admission into an introductory work like the present.

The method of analysing an equation proposed at page 127, and so fully illustrated in the closing chapter, and by which the most laborious part of Sturm's theorem is dispensed with, or rather exchanged for another operation, is I think deserving of examination, if not of general adoption. In equations of the more advanced degrees considerable advantage will often be derived from the examination of the leading *vertical* rows of signs in Sturm's table, as well as of the *horizontal* rows; agreeably to the suggestions of the method adverted to. I have applied this method to the analysis of a very unpromising example at page 221; and have, for the purpose of comparison, exhibited, in full detail, the corresponding calculations by the direct method of STURM, in a NOTE at the end.

There are also two other NOTES at the close of the volume to which I would invite attention. The first contains a useful and, as far as I know, an entirely new application of the well-known formula of CARDAN, to the solution of a cubic equation. The expression I have deduced from that of CARDAN furnishes, in a convenient form, the remaining two roots of a cubic equation in terms of the single root previously developed; and it exhibits the forms of these roots whenever they happen to be imaginary. The NOTE which follows this is allied to it by similarity of purpose: it will be found to supply a commodius formula for the remaining roots of a biquadratic equation after two of them have been discovered.

In several other places throughout the work occasional remarks and inferences may perhaps be met with which have hitherto escaped the observation of algebraists. Minute additions may thus be made to our knowledge of the nature and constitution of numerical equations; but they are too unimportant to be more than thus cursorily adverted to in a preface. The pretensions of the book are indeed but very moderate; my sole object having been to supply what appeared to me, and what I know has also appeared to others, a much-wanted link between the elements of common algebra and the more recondite researches of LAGRANGE, BUDAN, FOURIER, and

Sturm, on the general theory and solution of equations. My former volume had a somewhat higher aim; and was designed rather to present a condensed view of these researches than to furnish the young analyst with a detailed exposition of all the more elementary parts of the subject, delivered in a manner which a very moderate knowledge of algebra would suffice to render intelligible.

In the new edition of the larger work, I hope to preserve whatever claims the book has been thought to have upon the attention of a more advanced class of students. The introductory topics being disposed of in the present volume, I shall be enabled to dwell with fuller detail, upon the more general theories, and more largely to illustrate the new methods in some higher equations of peculiar difficulty.

J. R. YOUNG.

Belfast College;
*Nov.* 25, 1841.

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### ERRATA.

Page 13, line 23, *omit the words* the decomposition of.
20 18, *for* member *read* number.
140 2 from bottom, *for* always, *read* often.

ON THE

## ANALYSIS AND SOLUTION

OF

# EQUATIONS OF THE THIRD AND FOURTH DEGREES.

## INTRODUCTION.

(ART. 1.) In the Treatise on Algebra, of which the present work is to be considered as the continuation, we have entered into the doctrine of Equations only so far as was necessary in order to preserve a due proportion among the elements which compose the fundamental principles of algebraical science. What is there delivered on this subject refers exclusively to equations of the first and second degrees; and suffices only for the earlier stages in a course of analytical study; and for the junior classes in places of mathematical education.

It has indeed been customary for algebraical writers, though aiming only at this amount of usefulness, to introduce into their elementary treatises the more advanced topics of this extensive subject; and to insert rules and practical directions for the treatment of equations of the third and fourth degrees. These rules, however, the invention of the older algebraists, are all so complicated in their expression, so numerically laborious in their operation, and in certain cases so unsatisfactory in their results, as to present to the young student an aspect peculiarly formidable and repulsive. The progress of the learner is thus often likely to be arrested in the midst of his initiatory course; and that too by the injudicious introduction of topics which do not necessarily lie in his way, but which more properly belong to a subsequent stage in the orderly development of the science.

These considerations prevailed with the author in his endeavours to compose an elementary book on algebra for the use of beginners; and they have also determined him thus to draw up, in a separate form, an exposition of the analysis and solution of the classes of equations just mentioned, delivered in the simplest form he could devise, and accompanied with that fulness of detail, and of practical illustration, generally so acceptable to a learner, and always so essential to facility of acquisition. The methods to be employed in this exposition are the products of modern discovery. They entirely supersede the complicated and unsatisfactory processes adverted to above; and while they are perfectly general in their application, comprehending all numerical equations whatever, they admit of such peculiar adaptation to the classes here to be discussed as to render the analysis and solution of a cubic or of a biquadratic equation, heretofore problems of such tedious length and difficulty, a matter of very easy and rapid accomplishment.

We cannot however enter upon the investigation of these problems without first developing some of the fundamental properties common to equations in general. We shall therefore devote a preparatory section to this object.

## SECTION I.

### *On the leading Properties in the Theory of Equations.*

(2.) In the present Section we propose to give a brief summary of the Theory of Algebraical Equations, for the purpose of exhibiting the more important general relations which connect every equation with its roots.

By the *roots* of an equation we are to understand all those values, whether real or imaginary, which, when substituted for the unknown quantity in it, verify the equation; that is, which really fulfil the condition implied in the equation.

In the theory to be unfolded in the subsequent chapters, we recognize *determinate* equations only; that is to say, equations containing but one unknown quantity. Groups of equations, involving as many unknown quantities as there are distinct and independent equations, do virtually come within this theory; because every such group may be reduced, by elimination, to so many separate determinate equations, each involving but one unknown quantity. (Algebra, art. 117.)

(3.) Moreover, as preparatory to this theory we shall consider the coefficients of the equations discussed, as unencumbered with radical signs; and in general as still further simplified by the removal of the coefficient from the leading or highest power of the unknown quantity, a removal which may always be effected by dividing every term by that coefficient. After this preparation we shall regard our equations as free from every special restriction: that is to say, the coefficients which enter them may be either positive or negative, integral or fractional; or indeed any of them may be zero.

In this latter case, that is when one ormore of the terms of an equation are absent, the equation is called an *incomplete* equation;

and in bringing such equations within the general theory, it is usually necessary to give them the complete form, by the actual insertion of the absent terms with zero coefficients. Thus the incomplete cubic equation

$$x^3 - 6x + 10 = 0$$

is rendered complete by the insertion of $0x^2$, and may be written

$$x^3 + 0x^2 - 6x + 10 = 0.$$

(4.) It is proper further to observe that, in discussing the general theory, the equations are understood to be not only in their complete form, as respects the full complement of their terms, but also as regards the orderly arrangement of those terms, which arrangement requires that the significant terms of the equation should all be disposed on one side of the sign of equality, in the order of their dimensions, and that zero merely should occupy the other side. Hence a properly-arranged complete equation of the $n$th degree is thus expressed,

$$x^n \ldots . + A_3 x^3 + A_2 x^2 + Ax + N = 0.$$

It is sometimes, however, found convenient to arrange the terms of the *polynomial*, on the left of the sign of equality, in the *ascending* order of the powers, instead of in the descending order: thus

$$N + Ax + A_2 x^2 + A_3 x^3 + \ldots . x^n$$

(5.) The symbols $A, A_2, A_3$, &c., here employed, are merely arbitrary marks to represent the coefficients; and N stands for the absolute number, or quantity independent of $x$. This notation for the coefficients may at first seem rather cumbersome, because of the small subscribed figures attached to the letters. It is adopted, however, on account of its superior significance, as every coefficient, however isolated in a general investigation, thus carries with it a plain indication of the term to which it belongs. Thus $A_4$ belongs to the fourth power of the unknown, A to the first, and $A_n$ to the $n$th. And on the same principle we might write $A_0$ for N, as we should understand the complete term, into which this symbol enters, to be $A_0 x^0$; which, since $x^0 = 1$, is simply $A_0$, the term independent of $x$.

(6.) We have one more remark to make, and it is one which, on account of its importance, deserves the especial attention of the student. It is this, namely, that before developing the general theory of algebraical equations, it is essential either to prove, or to assume as a postulate, that *every such equation necessarily has a root.* In other words, that, whatever polynomial

$$x^n \ldots . + A_3x^3 + A_2x^2 + Ax + N$$

be proposed, there always exists, whether we can discover it or not, some value, real or imaginary, which, if substituted for $x$ in that polynomial, will reduce it to zero, thus fulfilling the condition

$$x^n \ldots . + A_3x^3 + A_2x^2 + Ax + N = 0.$$

This proposition is capable of rigid demonstration, although the method of proof—as is the case with most of the fundamental properties upon which the more important divisions of science are based—is somewhat long and difficult. As far as concerns *cubic* equations, however, the truth of the proposition is very easily established, without the aid of this general demonstration. In fact, the general expression for a root of a cubic equation, which has been actually exhibited in the *Algebra* (page 159), indisputably establishes the existence of such root. And as regards *biquadratic* equations, their actual solution, to be hereafter exhibited, as unequivocally proves the truth of the proposition with respect to *them.* Nothing, therefore, is to be taken for granted, without proof, within the scope of the present treatise; which does not extend its enquiries beyond equations of the fourth degree. The only reason for not actually exhibiting the general form of the roots of such an equation in this place is, that such a plan would conflict with an orderly exposition of our subject; and by postponing the general solution of a biquadratic equation to its proper place in such an exposition, we merely render the property referred to, and, of course, all that is dependent on it, hypothetical, till confirmed by our arrival at the solution adverted to.

The demonstration of the property, in all its generality, is given in the treatise on the *Theory and Solution of Equations in general.*

# CHAPTER I.

## ON THE CONSTITUTION OF EQUATIONS.

### PROPOSITION I. THEOREM.

(7.) 1. If a root of any equation

$$x^n \;.\;.\;.\;.\; + A_3x^3 + A_2x^2 + Ax + N = 0^* \;.\;.\;.\;.\; [1]$$

be $a$, then the polynomial which forms the first member of it will be divisible by $x - a$; and conversely:—

2. If the polynomial be divisible by the factor $x - a$, then will $a$ be a root of the equation.

1. Conceive the operation of dividing the polynomial by $x - a$ to be performed. Then it is plain, since the divisor contains $x$ only in the first power, that this operation will not terminate, so long as we continue to get a remainder involving $x$. In other words, the process will go on till we arrive at a remainder independent of $x$; and it is equally plain that none but positive integral powers of $x$ can enter the quotient.

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[*] The student must not forget that, agreeably to what has been already premised (5), the letters $A, A_2, A_3$, &c., are merely general symbols for the coefficients; which coefficients may be either positive or negative, integral or fractional, or even zero; so that we assume no hypothesis as to the signs, nor yet as to the numerical values of the coefficients by representing them by these symbols, preceded by the sign *plus;* which sign merely performs the office of connecting the several terms of the polynomial together, and implies nothing as to the actual signs of the coefficients in any proposed case.

Let the remainder be called R, and the quotient Q: then we shall have the identical equation

$$x^n \ldots + A_3 x^3 + A_2 x^2 + Ax + N = (x - a) Q + R.$$

Now if $a$ be a root of the equation, the polynomial, or, which is the same thing, the right-hand member of this identity, will be reduced to zero, when $a$ is put for $x$. But the substitution of $a$ for $x$ reduces that member to R: hence, if $a$ be a root of the equation, R must be zero, that is, $x - a$ will accurately divide the polynomial without leaving any remainder.

2. Conversely: let $x - a$ divide the polynomial without remainder; that is, let

$$x^n \ldots + A_3 x^3 + A_2 x^2 + Ax + N = (x - a) Q$$

where the quotient Q of this division contains none but positive integral powers of $x$, Put $a$ for $x$; and the second member of this equation is reduced to zero, and consequently the first member, which is identical to it, is reduced to zero. Therefore $a$ is a root of the equation [1].

### PROPOSITION II. PROBLEM.

(8.) To determine the quotient and remainder arising from dividing a polynomial of the form

$$A_n x^n \ldots + A_3 x^3 + A_2 x^2 + Ax + N$$

by any binomial of the form $x - a$.

The *remainder* is very easily found. For, as before, calling the quotient Q and the remainder R, we have

$$A_n x^n \ldots + A_3 x^3 + A_2 x^2 + Ax + N = (x - a) Q + R$$

and putting $a$ for $x$ we have

$$A_n a^n \ldots + A_3 a^3 + A_2 a^2 + Aa + N = R$$

the remainder sought; and which we see is obtained by merely substituting $a$ for $x$ in the proposed polynomial.

To find the quotient it is necessary to observe that as the divisor is an expression of the first degree, $x-a$, the quotient must be a polynomial a unit lower in degree than the proposed: if, for instance, the proposed be of the fourth degree, as

$$A_4x^4+A_3x^3+A_2x^2+Ax+N$$

the quotient Q will be of the third degree, or of the form

$$A'_3x^3+A'_2x^2+A'x+N'$$

so that we shall have to determine the values of the coefficients $A'_3$, $A'_2$, $A'$, and the value of $N'$, from the following identical equation, viz.

$$A_4x^4+A_3x^3+A_2x^2+Ax+N=$$
$$(x-a)(A'_3x^3+A'_2x^2+A'x+N')+R$$

The second member of this, upon executing the multiplication, is

$$A'_3x^4+(A'_2-aA'_3)x^3+(A'-aA'_2)x^2+(N'-aA')x-aN'+R$$

and, as this is identical to the first member, we have, by equating the coefficients of like powers of $x$, (Algebra, art. 121,)

$$A'_3=A_4$$
$$A'_2-aA'_3=A_3 \quad \therefore \; A'_2=A_3+aA'_3$$
$$A' -aA'_2=A_2 \quad \therefore \; A' =A_2+aA'_2$$
$$N' -aA' =A \quad \therefore \; N' =A +aA'$$

Moreover,

$$R -aN' =N \quad \therefore \; R =N+aN'$$

It thus appears that the coefficients of the quotient may all be rapidly deduced by a succession of uniform and very easy operations from the coefficients of the proposed polynomial; and that if these operations be extended to the absolute member N, the remainder R will also be obtained.

The first coefficient in the quotient is the same as the first in the dividend; and the uniform operations alluded to, by which the following coefficients are successively educed, are these, viz. each coefficient, as it is discovered, is multiplied by $a$; and the product added to the corresponding coefficient in the dividend: the result is the next coefficient of the quotient.

(9.) The best method of conducting these operations will be to arrange the given coefficients horizontally, and then to proceed as here directed.

### EXAMPLES.

1. Required the quotient arising from the division of

$$7x^4 - 13x^3 + 4x^2 + 6x - 12$$

by $x - 3$

Coefficients of the proposed

$$\begin{array}{rrrr} 7 & -13 & +\ 4 & +\ 6^{*} \\ & 21 & +24 & +84 \\ \hline & 8 & +28 & +90 \end{array}$$

Hence the quotient is

$$7x^3 + 8x^2 + 28x + 90.$$

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[*] Although the steps by which the coefficients of the quotient are reached in this method of proceeding, are so simple and obvious as almost to preclude the possibility of misapprehension; yet, for the sake of the beginner, it may be as well to formally describe them. 1. Having arranged the given coefficients in their order we multiply the *first* by 3, and place the product, 21, under the *second*; we then draw the horizontal line, and adding the 21 to the — 13, place the sum, 8, below this line. 2. We now multiply the 8 by the same 3, placing the product 24 under the next coefficient, above the line just drawn, and the sum of this and the 4 over it, that is 28, below the line, as before; and so on to the end.

2. Required the remainder in the foregoing example.

The remainder is got by joining the — 12 to the above horizontal row of coefficients, and continuing the work up to it: thus the whole process for finding both quotient and remainder will stand thus

$$\begin{array}{r} 7 - 13 + 4 + 6 - 12 \\ 21 + 24 + 84 + 270 \\ \hline 8 + 28 + 90 + 282 \end{array}$$

The remainder, therefore, is + 282

3. Required the quotient and remainder arising from the division of

$$x^5 + 7x^4 + 3x^3 + 17x^2 + 10x - 14$$

by $x - 4$

$$\begin{array}{r} 1 + 7 + 3 + 17 + 10 - 14 \\ 4 + 44 + 188 + 820 + 3320 \\ \hline 11 + 47 + 205 + 830 + 3306 \end{array}$$

hence the quotient is

$$x^4 + 11x^3 + 47x^2 + 205x + 830$$

and the remainder is + 3306

4. Required the quotient and remainder arising from dividing

$$5x^4 - 7x^3 - 4x^2 + 6x - 13$$

by $x + 6$, where $a = -6$

$$\begin{array}{r} 5 - 7 - 4 + 6 - 13 \\ -30 + 222 - 1308 + 7812 \\ \hline -37 + 218 - 1302 + 7799 \end{array}$$

Hence the quotient is

$$5x^3 - 37x^2 + 218x - 1302$$

and the remainder 7799.

5. Required the quotient and remainder resulting from the division of

$$3x^6 - 7x^4 + 4x^3 - x - 45624$$

by $x + 5$

Supplying the vacant terms in this incomplete expression we have

$$\begin{array}{r}
3 + 0 - 7 + 4 + 0 - 1 - 45624 \\
-15 + 75 - 345 + 1705 - 8525 + 42630 \\
\hline
-15 + 69 - 341 + 1705 - 8526 - \phantom{0}2994
\end{array}$$

Hence the quotient is

$$3x^5 - 15x^4 + 69x^3 - 341x^2 + 1705x - 8526$$

and the remainder $-2994$

### PROPOSITION III. THEOREM.

(10.) Every equation has as many roots as there are units in the exponent denoting its degree; that is an equation of the $n$th degree

$$x^n \dots + A_3x^3 + A_2x^2 + Ax + N = 0$$

has $n$ roots.

In order to demonstrate this important proposition it is necessary to admit that every equation has at least *one* root, a truth upon which we have already commented at (6). Let $a$ represent this root: then the first member of the proposed equation is divisible by $x - a$ (Prop. I) the quotient being of the form

$$x^{n-1} \dots + A'_2x^2 + A'x + N'$$

that is, we shall have

$$x^n \dots + A_3x^3 + A_2x^2 + Ax + N = (x-a)(x^{n-1} \dots + A'_2x^2 + A'x + N')$$

Now, as every equation has a root, there necessarily exists some value of $x$ which will satisfy the condition

$$x^{n-1} \ldots\ldots + A'_2x^2 + A'x + N' = 0$$

Let this value be $a_2$: then this second polynomial will be divisible by $x-a_2$, giving a quotient of the form

$$x^{n-2} \ldots\ldots + A''x + N'' = 0$$

so that

$$x^{n-1} \ldots\ldots + A'_2x^2 + A'x + N' = (x-a_2)(x^{n-2} \ldots\ldots + A''x + N'')$$

and consequently

$$x^n \ldots + A_3x^3 + A_2x^2 + Ax + N = (x-a)(x-a_2)(x^{n-2} \ldots + A''x + N'')$$

In like manner, because every equation has a root, there exists a value $a_3$ that will satisfy the condition

$$x^{n-2} \ldots\ldots + A''x + N'' = 0$$

so that this third polynomial is divisible by $x-a_3$, giving a quotient of a degree lower, that is of the degree $n-3$.

This last quotient is, by the same reasoning, divisible by a new binomial $x-a_4$, furnishing a new quotient of the degree $n-4$. And thus are these successive divisions practicable till the polynomial is at length reduced to the degree $n-n$, or 0; that is till it is absolutely exhausted; and which we see it must be after $n$ divisions. Consequently the proposed polynomial is made up of $n$ simple binomial factors; that is

$$x^n \ldots\ldots + A_3x^3 + A_2x^2 + Ax + N = (x-a)(x-a_2)(x-a_3) \ldots\ldots (x-a_n)$$

As the second member of this identity is reduced to zero for *either* of the following values of $x$, viz.

$$x=a \quad x=a_2, x=a_3, \ldots\ldots x=a_n$$

it follows that the equation

$$x^n \ldots\ldots + A_3x^3 + A_2x^2 + Ax + N = 0$$

has these $n$ roots.

(11.) This proposition makes known to us the constitution of every algebraic equation with one unknown quantity. It shows that when the terms of such an equation are all arranged on one side, the polynomial we thus get is composed of as many simple binomial factors as there are units in the exponent of the highest power of the unknown quantity. The discovery of these factors would be the discovery of the roots of the equation, since these are at once given by equating the several factors each to zero. Or if we could find the roots we could at once pronounce upon the composition of the polynomial forming the significant side of the equation; since each root, connected with a changed sign to the unknown symbol, would furnish all the simple factors of which the polynomial is composed. By the *solution of an equation* is meant the determination of all its roots. The solution of an equation, therefore, is equivalent to the decomposition of the polynomial which is equated to zero into its component simple factors. If we could solve every algebraical equation, we could thus discover the composition of every polynomial containing but one unknown quantity; for we should only have to equate the polynomial to zero, and then to solve the equation thus formed. All *numerical equations*, that is to say, equations whose coefficients are given numbers, and not unknown letters, *can* be solved by methods of unfailing certainty; so that the decomposition of every rational and integral polynomial, involving but one unknown quantity, with numerical coefficients, can be actually decomposed into its simple elements.[*]

If such a polynomial be but of the second degree, then the common rule for quadratics will effect the decomposition, and that too whether the coefficients be numeral or literal. For example, if it were required to decompose the quadratic expression $x^2 + 6x - 55$ into its simple factors, we should merely have to solve the equation

$$x^2 + 6x - 55 = 0$$

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[*] See the Author's Treatise on the *Theory and Solution of Equations in general.*

the roots of which we should find to be

$$x = 5 \text{ and } x = -11$$

showing that the factors of the proposed expression are $x - 5$ and $x + 11$, that is

$$x^2 + 6x - 55 = (x - 5)(x + 11)$$

(12.) It will be observed that the above theorem does not affirm that an equation of the $n$th degree has $n$ roots all differing from one another in value. It only proves that there are $n$ simple factors in the significant side of such an equation, by equating each of which to zero we are furnished with $n$ roots for the equation. But any number of these factors, and of these roots, may turn out to be equal to one another.

It may be further remarked here that, whatever difficulty may attend the actual decomposition of an equation into its simple factors, the reverse operation of constructing the equation whose simple factors or roots are given, is very easily performed; nothing more being necessary than the mere multiplication of the proposed factors together, and the equating of their product to zero. Thus, if it be required to form the equation whose roots are $1, -1$, and $2$ we shall merely have to multiply the simple factors $(x - 1)$, $(x + 1)$ and $(x - 2)$ together, and to equate the product to zero. The required equation is therefore

$$x^3 - 2x^2 - x + 2 = 0$$

In like manner the equation which has $-1, -1, 4$ and $9$ for its roots is found, by multiplying together the factors $(x + 1)$, $(x + 1)$, $(x - 4)$ and $(x - 9)$, to be

$$x^4 - 11x^3 + 11x^2 + 59x + 36 = 0$$

(13.) The learner will plainly perceive that it would be impossible to give a conclusive demonstration of the foregoing proposition without first assuming that every equation must have *one* root. And this assumption being made the theorem here announced is so obviously involved in it that very little development is really necessary.

To say that an equation has a root, is to say that the polynomial forming its first member has a simple factor, $x-a$. The remaining factor, a polynomial of a degree a unit lower than the proposed, has in virtue of the same principle, a simple factor, $x-a_2$; so has the factor, of a degree two units lower, which in combination with the two simple factors $x-a, x-a_2$, make up the proposed polynomial; and so on. And thus the polynomial is wholly made up of simple factors.

(14.) There is a class of equations called *binomial equations*: these consist of but two significant terms, their general form being

$$x^n \pm \mathrm{N} = 0$$

A root of any equation belonging to this class is always easily exhibited: thus taking the upper sign we have for $x$ the expression $x=(-\mathrm{N})^{\frac{1}{n}}$; and taking the lower sign we have $x=\mathrm{N}^{\frac{1}{n}}$; so that in a polynomial of this form we can always adduce *one* simple factor by which it is divisible.

For the purpose of simplifying the root put $a^n$ for N; then a root of $x^n-a^n=0$ will be $a$; so that $x-a$ is necessarily a factor of the first member, that is $x^n-a^n$ is always divisible by $x-a$ whatever be the value of $n$. Also, if $n$ be an *even* integer, $-a$ will be a root as well as $+a$: hence in this case $x^n-a^n$ will be divisible both by $x-a$ and $x+a$.

A root of $x^n+a^n=0$ will be $-a$, *provided n be an odd integer;* but neither $-a$ nor $+a$ will be a root if $n$ be even. Hence $x^n+a^n$ is always divisible by $x+a$ if $n$ be odd, but not else; and whatever be $n$ the expression is never divisible by $x-a$.

It is easy to find the quotients due to these divisions from what has been delivered in proposition II.) Thus the quotient arising from the division of $x^n-a^n$ by $x-a$ is found as follows

$$\begin{array}{l}
1+0+0+0+0 \;\ldots\ldots +0 \\
\quad\; a+a^2+a^3+a^4 \;\ldots\ldots\; a^{n-1} \\
\hline
\quad\; a+a^2+a^3+a^4 \;\ldots\ldots\; a^{n-1}
\end{array}$$

The quotient is therefore

$$x^{n-1}+ax^{n-2}+a^2x^{n-3}+a^3x^{n-4}+a^4x^{n-5}+\ldots a^{n-1}$$

In like manner, *when n is even,*

$$\frac{x^n-a^n}{x+a}=x^{n-1}-ax^{n-2}+a^2x^{n-3}-a^3x^{n-4}+a^4x^{n-5}-\ldots-a^{n-1}$$

(15.) It is pretty obvious that besides the $n$ simple factors, into which we have seen a polynomial of the $n$th degree is decomposable, no other factor can enter into that polynomial; but it is perhaps better to give a formal proof of this truth than to leave it to be immediately inferred from the preceding investigation.

### PROPOSITION IV. THEOREM.

(16.) An equation of the $n$th degree can have no more than $n$ roots.

It has been just proved that an equation of the $n$th degree may be exhibited under the following form, viz.

$$(x-a)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0\ldots\ldots[1]$$

If this equation had any root $\alpha$ different from the roots

$$a,\ a_2,\ a_3\ldots\ldots a_n\ldots\ldots[2]$$

then upon putting $\alpha$ for $x$ the first member would be reduced to zero; and consequently one, at least, of the factors which form it would be reduced to zero. But this is obviously impossible, except $\alpha$ be identical to one of the quantities [2]. Hence the equation can have no more roots than the $n$ roots [2].

(17.) It is obvious that if a root $a$ of the equation [1] be discovered, the division of the first member by $x-a$ will give for quotient the polynomial involving the remaining factors; so that equating this to zero we shall have the *depressed equation*, as it is called, containing all the other roots. This depressed equation is readily deduced by the method explained at page 9, as in the following examples.

1. One root of the biquadratic equation

$$x^4+9x^3+9x^2-41x-42=0$$

is 2; required the depressed equation involving the three remaining roots

$$\begin{array}{r} 1+9+\ 9-41 \\ 2+22+62 \\ \hline 11+31+21 \end{array}$$

Hence the depressed equation is

$$x^3+11x^2+31x+21=0$$

2. A root of this latter equation is $-1$: required the quadratic containing the other two roots.

$$\begin{array}{r} 1+11+31 \\ -\ \ 1-10 \\ \hline 10+21 \end{array}$$

Therefore the required quadratic is

$$x^2+10x+21=0$$

the roots of which are $-3$ and $-7$; so that the four roots of the original equation are 2, $-1$, $-3$ and $-7$, the polynomial forming its first member being made up of the factors

$$(x-2),\ (x+1),\ (x+3) \text{ and } x+7$$

that is

$$x^4+9x^3+9x^2-41x-42=(x-2)(x+1)(x+3)(x+7)$$

It is thus easy to see how from knowing *one* root of a *cubic* equation we can obtain the depressed quadratic, and thence the other two roots; and how from knowing *two* roots of a *biquadratic* equation we may effect the same thing with respect to it.

### PROPOSITION V. THEOREM.

(18.) If two numbers or quantities, when separately substituted for the unknown quantity in an equation, give results with *opposite signs* then one real root of that equation, at least, must lie between the quantities so substituted.

Let $\alpha$ be a quantity which, when substituted for the unknown in the proposed equation, gives for the value of the significant side a *plus* result; and let $\beta$ when so substituted give a minus result: it is required to prove that one root at least of the equation is in value between $\alpha$ and $\beta$.

Let the equation, with its component factors exhibited, be

$$(x-a)(x-a_2)(x-a_3)\ldots\ldots(x-a_n)=0$$

By hypothesis the first member gives a *plus* product when $\alpha$ is put for $x$. This product necessarily continues plus for every other substitution which preserves the signs of the several factors unchanged; and more, it still continues plus though the signs of certain factors *do* change, provided the number of such factors be *even*. All this is plain, because the sign of a product can never change so long as the signs of the factors which form it are preserved, and the change of sign of two, four, or of any even number of these factors has no effect on the sign of the product. Hence, as the substitution of $\beta$ for $x$ *does* produce a change of sign in the product, it follows that this substitution must produce a change of sign in *one*, *three*, *five*, or some *odd* number of the component factors. One factor at least, therefore, as for instance the factor $(x-a)$, must take opposite signs when $\alpha$ and $\beta$ are successively substituted in it for $x$. When the root $a$ is substituted this factor is zero; hence the quantity which when substituted renders it plus, a result greater than zero, must be *greater* than this root, and the quantity which renders it minus must be *less* than the root; that is, the root must lie between the two.

(19.) It thus appears that when two numbers, substituted successively for $x$ in an equation, give results with *like* signs, either no root at all exists between those numbers, or else there are an even number of roots between them. But when the results have unlike signs, one root necessarily lies between the numbers substituted. There may be more; but their number must be odd.

By way of illustration let us take the equation adduced in last proposition, viz., the equation

$$x^4+9x^3+9x^2-41x-42=0 .$$

If 0 be substituted for $x$ the result is negative, viz. $-42$; and if 10 be substituted the result is obviously positive. Hence an odd number of roots, that is, either one or three, are *positive*, and below 10 in value. If $-10$ instead of $+10$ be put for $x$ the result is in like manner positive. Hence there must be an odd number of roots between 0 and $-10$, that is, an odd number of negative roots numerically below $-10$.

(20.) It follows from the proposition just established, that if the roots of an equation be all *imaginary*, that is, if no real number exist that can reduce the first member to zero; then, substitute in that equation whatever numbers we may for the unknown quantity, the results which arise will always have the same sign: for if contrary signs ever occurred, their occurrence would indicate the existence of a *real root* between the numbers leading to such results.

(21.) We may easily discover what this invariable sign is, as also what is the general character of equations whose roots are all imaginary. For we may write the first member of the proposed equation

$$x^n \ldots\ldots + A_3 x^3 + A_2 x^2 + A x + N = 0$$

thus, viz.

$$x^n \left(1 \ldots\ldots + \frac{A_3}{x^{n-3}} + \frac{A_2}{x^{n-2}} + \frac{A}{x^{n-1}} + \frac{N}{x^n}\right)$$

which expression is to be such that whatever numbers we substitute for $x$ the sign of the result is invariably the same. Now the larger the numbers substituted the more insignificant do the fractions within the parenthesis become; and if a number infinitely large be substituted these fractions severally vanish, and the whole result is reduced to $\infty^n$ or $(-\infty)^n$, according as our infinite number, $\infty$, is positive or negative.

If $n$ be odd then $\infty^n$ is *positive* and $(-\infty)^n$ *negative*; that is, the results have *opposite signs*. Therefore $n$ cannot be odd. The degree of the equation must therefore be *even*. In this case the results of the foregoing substitutions are *positive*; this therefore is the sign that invariably results from the substitution of any member for $x$ in an equation whose roots are all imaginary.

Moreover, if 0 be substituted for $x$, the polynomial is reduced to its final term N, the sign of this final term therefore is in such equations invariably *positive*.

The following inferences are thus established, viz.

1. Every equation of an odd degree has at least *one* real root.
2. And so has every equation of an even degree, provided the sign of the final term be negative.

It is the object of the two propositions next following to amplify these inferences.

### PROPOSITION VI. THEOREM.

(22.) Every equation of an odd degree has at least one r root, of which the sign is contrary to that of the final term of equation.

1. First, let the final term N be *negative;* and let the component factors of the first member of the equation be

$$(x-a)\,(x-a_2)\,(x-a_3)\,\ldots\,(x-a_n)$$

Now, however great the greatest of the roots $a, a_2$, &c. may be, a positive number K may always be assigned that shall exceed it. Conceive such a number substituted for $x$ in the foregoing expression; then all the *real* factors of it will be positive, and since the product of the imaginary factors, should any enter, are positive too (21) it follows that the entire product is *positive.*

But if 0 be substituted for $x$, the entire product will be reduced to its final term N, which by hypothesis is *negative.* Hence a root exists between 0 and the positive number K, that is, when N is negative the equation necessarily has a positive root.

2. Next, let N be positive; then, however small be the smallest root of the equation, a negative number $k$, still smaller, may be assigned; that is to say such a number as will render all the real factors *negative.*

Now these real factors must be odd in number, because the imaginary factors, if any, are necessarily even in number as proved at (21).

Hence the product of these negative factors must be negative, and as the product of the imaginary factors, if any, are positive (21) it follows that the entire product is *negative.*

But if 0 be substituted for $x$ the entire product will be reduced to the final term N, which is *positive;* consequently a root of the equation lies between 0 and the negative number $k$; that is, when N is positive the equation necessarily has a negative root.

### PROPOSITION VII. THEOREM.

(23.) Every equation of an even degree with its final term negative has at least two real roots, one positive and the other negative.

It is proved in the reasoning marked 1 of last proposition, that *every equation* whose last term is negative has a *positive* root: for that reasoning is independent of every condition as to the *degree* of the equation.

And as in the case before us the real factors of the equation must be even in number, it follows that if a negative number $k$, smaller than the smallest root, be put for $x$ in them all, that is, a number that will render all these factors negative, their product will be *positive*; and consequently (21) the entire product will be *positive*. But if 0 be put for $x$ the entire product, which is then reduced to the final term, is by hypothesis negative. Hence there exists in the equation a *negative* root between 0 and the negative number $k$.

### PROPOSITION VIII. THEOREM.

(24.) If the signs of the alternate terms in a complete equation be changed, the signs of all the roots of it will be changed.

Let the equation be

$$x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + A_{n-3}x^{n-3} + \ldots = 0 \ldots [1]$$

and with its alternate signs changed

$$x^n - A_{n-1}x^{n-1} + A_{n-2}x^{n-2} - A_{n-3}x^{n-3} + \ldots = 0 \ldots [2]$$

First, let $n$ be even; then all the odd powers only in [1] appear in [2] with a change of sign. Hence, if any quantity be substituted for $x$ in [1] the result will be the same as if the same quantity with a changed sign were substituted in [2]; since none but odd powers of a quantity are affected by a change of sign in that quantity.

Let $a$ be any root of [1]; then $a$ put for $x$ in [1] reduces the first member to zero. But, as just seen, $-a$ put for $x$ in [2] gives the same result; that is, it reduces the first member of

[2] to zero; that is, $-a$ is a root of [2]. Hence, for every root $a$ in [1] there is an equal root with opposite sign, $-a$, in [2]; so that the roots of [2] are the same as those of [1] only the signs of them are opposite.

Next let $n$ be odd; then all the even powers only in [1] appear with changed signs in [2]. Hence if $a$ be put for $x$ in the former, and $-a$ for $x$ in the latter, the several terms of [2] will obviously be those of [1] with all their signs changed. But if $a$ be a root of [1] the aggregate of these terms will be zero; and therefore also zero though the signs of them be all changed, which we see they will be by the substitution of $-a$ in [2]. Hence $-a$ is a root of [2]. Consequently, for every root in [1] there is an equal root in [2] with opposite sign.

By means of the foregoing general property we can always convert the positive roots of every equation into negative, and the negative roots into positive; nothing more being requisite for this purpose than the changing of the signs of the alternate terms of the equation. For example, it has been seen (17) that the roots of the equation

$$x^4+9x^3+9x^2-41x-42=0$$

are 2, $-1$, $-3$ and $-7$. By changing the signs of the alternate terms the equation becomes

$$x^4-9x^3+9x^2+41x-42=0$$

of which the roots are $-2$, 1, 3, and 7, as is easily proved by substituting each of these numbers for $x$ in the first member.

### PROPOSITION IX. PROBLEM.

(25.) To determine the composition of the coefficients of an equation and the relations which they bear to the roots.

At art. (17) we have shown how, when a known root enters an equation, we may eliminate that root and thus depress the equation a degree lower. We may as readily perform the reverse operation; and by introducing a proposed root elevate the equation a degree higher. By generating equations in this manner, that is, by the successive introduction of new roots, we should obviously discover in what way these roots go to the formation of the several coefficients. But before proceeding to general reasonings let us illustrate, by a particular example, the method of performing the reverse operation alluded to; that is of introducing new roots into a given equation.

We shall take the quadratic equation

$$x^2+10x+21=0$$

to which the equation quoted above was reduced at (17) by the successive elimination of the roots 2 and $-1$; and we shall seek to restore the equation to its original state by the re-introduction of these roots. A mere glance at the operation at (17) is sufficient to suggest the reverse process now to be employed. The coefficients of the cubic, 11, 31, 21, are evidently to be derived by multiplying those of the given quadratic by the root $-1$, *with its sign changed*, placing the products in a row under the latter, commencing at the second, 10, and then adding.

In like manner the coefficients of the next advanced equation, the biquadratic, are obtained by multiplying in the same way the coefficients just deduced by the next, 2, *with its sign changed*, arranging the products and adding as before. Here is the work, the roots introduced being exhibited with change of sign on the right:

$$\begin{array}{rrrrrl}
1 & +10 & +21 & & & (1 \\
 & 1 & +10 & +21 & & \\
\hline
1 & +11 & +31 & +21 & & (-2 \\
 & -2 & -22 & -62 & -42 & \\
\hline
1 & +9 & +9 & -41 & -42 &
\end{array}$$

Hence the equation involving the two roots $-1$ and $2$, as well as those of the given quadratic, is

$$x^4 + 9x^3 + 9x^2 - 41x - 42 = 0$$

And it is evident that by proceeding continually in this manner, as many roots as we please may be introduced.

Let us now take a *general* quadratic equation, the roots being represented by $a$, $a_2$. The first member of this quadratic, being produced from the factors $(x - a)$, $(x - a_2)$, is $x^2 - (a + a_2)\,x + a\,a_2$; where the coefficient of the second term is the sum of the roots with their signs changed, and the last term is their product. Introduce the new root $a_3$; then as above

$$\begin{array}{llll}
1 & -(a + a_2) & +\,a\,a_2 & (-a_3 \\
 & -a_3 & +\,(a\,a_3 + a_2 a_3) & -a\,a_2\,a_3 \\
\hline
1 & -(a + a_2 + a_3) & +\,(a\,a_2 + a\,a_3 + a_2 a_3) & -a\,a_2\,a_3
\end{array}$$

Here, as before, the coefficient of the second term is the sum of all the roots with their signs changed, the next coefficient the sum of all their products taken two and two, and the next the product of all three with their signs changed.

It is evident that by introducing another root $a_4$, that is by employing, as above, a new multiplier $-a_4$, the composition of our result would be similar; that is, the second coefficient would be the sum of all the roots with their signs changed, the next the sum of their products two and two; the next the sum of their products three and three, their signs being changed; and the next the product of all four.

And from the uniform manner in which new roots $a_5$, $a_6$, &c. would be successively introduced, it is plain that this law of the formation of the several coefficients is universal: that is in any equation

$$x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + A_{n-3}x^{n-3} \ldots + A_3x^3 + A_2x^2 + Ax + N = 0$$

the composition of the coefficients is as follows :

$A_{n-1}$ is the sum of the roots when their signs are changed,
$A_{n-2}$ is the sum of the products of every two . . . . .
$A_{n-3}$ . . . . . . . . . . . . . three . . . . .
.
.
.
.
.
$A_3$ . . . . . . . . . . . . . . $n-3$ . . . . .
$A_2$ . . . . . . . . . . . . . . $n-2$ . . . . .
$A$ . . . . . . . . . . . . . . $n-1$ . . . . .
$N$ . . . . . . . . . . . . . $n$ . . . . .

In the *third, fifth, seventh,* &c. coefficients, it is obviously indifferent whether we regard the signs of the roots changed or not; as these consist of the products of an *even* number of factors.

Several inferences may be deduced from this law, the most interesting of which are.

1. That if the coefficient of the second term of an equation be 0, the sum of the positive roots must equal the sum of the negative roots; and

2. That every root is a divisor of the last term, N, of the equation.

### PROPOSITION X. THEOREM.

(26.) If the coefficient of the leading term of an equation be unity, and all the other coefficients whole numbers, that equation cannot have a fractional root.

This proposition may be very easily proved by means of the two following axiomatic principles, viz.:

1. If a fraction in its lowest terms be added to a whole numb ·, the improper fraction resulting must also be in its lowest ter . For as one part of the compound numerator will be divisible b a

factor of the denominator, and the other part, not the whole, cannot be divisible.

2. If a fraction in its lowest terms be multiplied by another having the same denominator, the product will be a fraction in its lowest terms. For if the numerator had any factor common to the denominator, the same common factor would exist also in one or other of the original fractions, which is impossible, because these are in their lowest terms.

Now suppose that an equation under the conditions announced can have a fractional root. Then in order to eliminate this root and get the depressed equation, we must proceed as at (17); that is, we must multiply the supposed root by the first coefficient, 1, and add the product to the second coefficient; this sum must in like manner be multiplied by the same root, and the product added to the third coefficient, and so on. The product which falls under the absolute number N, will of necessity be equal to it, and therefore furnish no remainder. But from the preceding principles all these products are fractions in their lowest terms, so that the product which falls under the whole number N cannot be equal to it. Hence, under the conditions proposed, no equation can have a fractional root.

It follows, therefore, that when the first coefficient is unity, and all the others whole numbers, those roots of the equation which are not whole numbers must be interminable decimals; or, as they are frequently called, *incommensurable*. And if there be one such root, there must necessarily be another; because the sum of all the roots, that is the coefficient of the second term, (25) is a whole number.

### *The Rule of Signs Harriot, commonly called the Rule of Descartes.*

(27.) There is a certain connexion between the number of positive and negative roots of an equation, and the signs merely of the terms of the polynomial forming its first member.

This connexion is such, that if we are only assured that the roots of the equation are all *real*, we can immediately ascertain, from the character of the signs which succeed one another, how many of these roots are positive, and how many are negative. We seldom, however, know much about the nature of the roots, as to real and imaginary, before actually analysing the equation; and thus the rule of Descartes, providing for the entrance of imaginary roots, only states the number of positive roots, and the number of negative roots, which those of the proposed equation cannot *surpass.* The rule is enunciated as in the following proposition.

### PROPOSITION XI. THEOREM.

(28.) An equation cannot have a greater number of *positive* roots than there are *changes* of sign from + to —, and from — to + in the series of terms forming its first member. Nor can it have a greater number of *negative* roots than there are *permanencies* or repetitions of sign in proceeding from term to term.

In order to demonstrate the truth of this rule, it will be necessary to show, 1. That if any polynomial be multiplied by a factor $x - a$, corresponding to a *positive* root, there will be at least one *change of sign* more in the result than in the original polynomial; and 2. That if it be multiplied by $x + a$, corresponding to a *negative* root, there will be at least one permanence of sign more in the result than in the original polynomial.

Let the signs of the proposed polynomial succeed one another in any order, as for instance, in the order

$$+ - - + - + + + - - +$$

And 1. Let the polynomial be multiplied by $x - a$; that is, let the root $a$ be introduced into the equation; then the sign of $a$ being +, the operation, conducted as already explained (25),
in which this plus is changed to *minus*, will give rise to a sec
row of signs, which will arrange themselves under the given r

commencing under the second of that row, and terminating one step beyond it. These two rows, and that due to the product resulting from them, are therefore as follow:

$$\begin{array}{cccccccccccc}
+ & - & - & + & - & + & + & + & - & - & + & \\
 & - & + & + & - & + & - & - & - & + & + & - \\
\hline
+ & - & \pm & + & - & + & \pm & \pm & - & \pm & + & -
\end{array}$$

We have written the ambiguous sign $\pm$ wherever the addition of *unlike* signs occurs, because in the absence of the quantities themselves, to which these signs belong, we cannot know which predominates.

It is plain from the manner in which the second row is formed, by changing one after another the signs of the first row, that in this and in every other arrangement, an ambiguity in the last row must always occur wherever there is a permanency in the first row; that is, there must be exactly as many ambiguities in the resulting row of signs as permanencies in the given row. In the proposed arrangement above, there are *four* permanencies, viz.:

$$- -,\ + +,\ + +,\ - -$$

and accordingly there are four ambiguities in the product; the other signs remaining just the same as in the given row, with the addition of the final sign which is superadded, and which is always unlike the final sign in the proposed.

These inferences are obviously quite general; so that, knowing the signs of any polynomial whatever, we can always write under them the signs in the product of that polynomial by the factor $x - a$, as far as these signs are determinable without knowing the actual values of the quantities employed; for we shall merely have to write an ambiguity for every permanency, copying the intermediate signs as they stand, and then to superadd a final sign unlike that with which the proposed row terminates. If, for example, the proposed row of signs were

$$+ - - + + + + - + - + +$$

the signs of the product would be

$$+ - \pm + \pm \pm \pm - + - + \pm -$$

In like manner, if the proposed row were

$$+ + - + - + - - - - + -$$

the signs of the product would be

$$+ \pm - + - + - \pm \pm \pm + - +$$

As therefore in passing from the multiplicand to the product the permanencies alone can suffer change, the variations continuing unaltered, these latter cannot possibly be *diminished* in number, however they may be increased by the change of permanencies into variations. Hence, without taking into account the superadded sign, there are at least as many variations of sign in the product as in the original polynomial.

Now this superadded sign necessarily introduces a variation; for if the proposed terminate with a variation, the superadded sign, being unlike the one immediately before it, introduces another variation; and if the proposed terminate with a permanency, then the superadded sign is immediately preceded by an ambiguity; and whatever this turns out to be for any given values, it must form a variation either with the superadded sign, or with the sign immediately before it; a variation is thus either actually *added*, or else an original permanency converted into one.

Hence an equation cannot have a greater number of *positive* roots than *variations* of sign in its first member.

2. Again, let us suppose a negative root $-a$ to be introduced, or that the given polynomial is multiplied by $x+a$. Whatever

be the signs in the resulting equation, we know (24) that by changing them alternately, we should convert the positive roots into negative and the negative into positive. But such changes would obviously convert the permanencies into variations, and the variations into permanencies; and it has just been proved that this changed equation cannot have a greater number of positive roots than variations. Hence the proposed cannot have a greater number of *negative* roots than *permanencies* of sign in its first member.

(29.) We immediately infer from this rule,

1. That when the signs of the first member are all positive, the equation cannot have a positive root.
2. When the signs are alternately positive and negative the equation cannot have a negative root.

But the *exact number* of positive or of negative roots in any proposed equation it is not in the power of the rule to discover. The possible occurrence of *imaginary roots* deprives it of this precision. It is, however, in certain cases effective in discovering the existence of such roots, as their entrance sometimes occasions discrepancies in the character of the signs incompatible with the existence of real roots, and which the rule of Descartes enables us to detect. These cases offer themselves in *incomplete* equations, when the terms on each side of the vacant place have *like* signs. For example, the character of the roots of the incomplete equation

$$x^3+Ax+N=0$$

when the coefficient A is *positive*, is readily discovered by the rule of signs; for restoring the absent term it may be written either

$$x^3+0x^2+Ax+N=0$$

or

$$x^3-0x^2+Ax+N=0$$

In the first of these there are no variations, and therefore by the rule no positive roots. In the second there is but one permanence, and therefore by the rule not more than one negative root. These conclusions are contradictory on the supposition of the roots being all real; we infer, therefore, that the proposed equation has imaginary roots; and therefore (21) a *pair* of such roots. The other root is real, of course (22).

If N be supposed negative instead of positive in the above equation, a similar discrepancy would be manifest; the first form of the equation would imply the existence of only one positive root at most; and the second form would forbid any negative roots.

But if the form of the equation had been

$$x^3 - Ax + N = 0$$

in which the coefficient of $x$ is negative, then the complete forms

$$x^3 + 0x^2 - Ax + N = 0$$

and

$$x^3 - 0x^2 - Ax + N = 0$$

whether N be positive or negative, would offer no discrepancy; because each would exhibit the same variations, and the same permanencies, so that we are unable to say whether imaginary roots enter the equation or not.

These applications of the rule of signs to incomplete equations, for the purpose of detecting the existence of imaginary roots, may be easily extended to the cases in which several terms are absent, whether the blanks occur singly or in groups. But we must refer for the discussion of these cases to the *Treatise on the General Theory and Solution of Equations*, second edition.

(30.) It remains that we show the rule to be infallible, when the roots of the equation under examination are known to be all real. This we may do as follows:

Let $n$ be the degree of the equation. Then obviously if $p$ the number of permanencies, and $v$ the number of variations must have $p + v = n$.

Let $p'$ be the number of negative roots, and $v'$ the number of positive roots, then also $p'+v'=n$

$$\therefore p+v=p'+v'$$

Now it has been proved that $p'$ cannot be greater than $p$, and that $v'$ cannot be greater than $v$; hence we must necessarily have

$$p=p' \text{ and } v=v'$$

Consequently, when the roots are all real, the number of positive roots will be exactly equal to the number of variations, and the number of negative roots to the number of permanencies.

It must be borne in mind, however, that whether the roots are all real or not, the equation is to be rendered *complete* before it is submitted to the rule of signs. The investigation of that rule plainly comprehends complete equations only; although, as it is easy to see, the first conclusion arrived at would be equally true if the signs originally proposed were those of an *incomplete* equation, taken in order as they actually occur. For by supplying the vacant places we should merely increase the number of ambiguities, and could not therefore *diminish* the number of variations already obtained. The number of permanencies, however, might be diminished by some of these new ambiguities turning out variations in particular cases, and thus the second conclusion might not hold.

For example, the equation

$$x^4-3x^2+5x-7=0,$$

disregarding the absent term $\pm 0x^3$, presents no permanence of sign. But it would be incorrect to infer from this that the equation has no negative root; for we know, from the sign of the last term, that it has at least *one* such root (23).

The rule, therefore, as expressed in the proposition requires, for its correct application, that the equation be first put in a complete form. In the present instance it should be written thus:

$$x^4 \pm 0x^3-3x^2+5x-7=0$$

which presents a permanence of sign, whether the zero coefficient be regarded as + or —.

Still, without making any condition as to whether the equation be complete or not, the rule may be so expressed as to apply infallibly to every equation. Thus modified, the rule of signs is as follows:

(31.) In any equation, complete or not, the number of positive roots cannot exceed the number of variations which the signs of the polynomial present; and the number of negative roots cannot exceed the number of variations which that polynomial *would present*, if $-x$ were substituted in it for $x$.

The substitution of $-x$ for $x$ obviously converts the second case of the rule into the first; which, as shown above, is not affected by the non-insertion of the wanting terms of an incomplete equation.

# CHAPTER II.

## ON THE TRANSFORMATION OF EQUATIONS.

(32.) WITHOUT knowing what the roots of an equation really are we can always introduce such changes into it as to bring about certain relations among the unknown roots, these changes having the effect of so changing the roots as to render the prescribed relations among them inevitable.

We may, for example, transform any proposed equation into another, which shall have all its real roots *positive;* or all negative; or which shall be those of the original equation equally increased or diminished, multiplied or divided. And when any of these changes are effected we shall always be able to determine the roots of the original equation from those of the transformed equation, provided we have any means of discovering these latter.

This theory of the transformation of equations has therefore an immediate bearing upon their practical solution; and on this account it is a branch of the general doctrine which was much studied and cultivated by the older algebraists. But in the present improved state of this department of analysis, into which such important practical facilities have been recently introduced, as to render most of these preparative transformations of but little moment, this theory need not be discussed at any length. The propositions following will amply suffice for the purposes of the present volume.

### PROPOSITION I. PROBLEM.

(33.) To transform an equation into another whose roots shall be given multiples, or submultiples of those of the proposed equation.

If the leading term in the proposed equation have a coefficient other than unity let all the terms be divided by it: then, in the resulting equation, the coefficient of the second term will be equal to the sum of the roots with changed signs; the next coefficient the sum of the products of the roots two and two; the next the sum of the products three and three, signs being changed; and so on (25). Hence for the roots to be $m$ times what they are at present ($m$ being whole or fractional) we must multiply the second term by $m$, the third by $m^2$, the fourth by $m^3$, and so on.

As the multiplication of all the terms of an equation by the same thing does not alter the roots, the transformed equation may now be multiplied by the coefficient of the leading term of the original; that is the leading coefficient, temporarily removed for the sake of simplifying the reasoning, might have remained without disturbing the conclusion.

As an example, let it be required to transform the equation

$$x^3-2x^2-x+2=0$$

whose roots are 1, 2, and $-1$, into another whose roots are three times as great.

Here the second, third, and fourth terms must be multiplied by 3, $3^2$, and $3^3$ respectively. The resulting equation is

$$x^3-6x^2-9x+54=0$$

the roots of which are 3, 6, and $-3$, as the substitution of each of these numbers for $x$ will prove.

Again, let the equation

$$x^3-28x+48=0$$

be transformed into another whose roots are only half as grea

This equation when rendered complete is

$$x^3\pm 0x^2-28x+48=0$$

Hence, multiplying the second, third, and fourth terms by $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{8}$ respectively, we have

$$x^3 - 7x + 6 = 0$$

for the required transformed equation.

The roots of this are 1, 2, and $-3$, as a trial of these numbers will show; hence the roots of the proposed are 2, 4, and $-6$.

(34.) The foregoing transformation is useful as a means of removing the coefficient from the leading term of an equation without introducing fractions. For if $m$ be this coefficient, and we transform the equation into another whose roots are $m$ times as great, the transformed equation will have the factor $m$ in all its terms: if, therefore, we divide by this factor the coefficient of the first term, will be removed, and no fractions will be introduced. The transformed equation thus simplified will therefore be at once obtained by expunging the coefficient of the first term of the proposed, preserving the coefficient of the second term as it is, multiplying the coefficient of the third by $m$, that of the fourth by $m^2$ and so on. The roots of the equation thus modified will be $m$ times the roots of the original.

(35.) Fractions may be removed from the coefficients of an equation by the same method. We have only to find a common multiple of the denominators of these fractions, and then to transform the equation into another, whose roots shall be those of the former multiplied by this common multiple. For instance, the equation

$$x^4 + \frac{1}{2}x^3 - \frac{1}{3}x^2 - \frac{1}{6}x + 10 = 0$$

is thus transformed into

$$x^4 + 3x^3 - 12x^2 - 36x + 12960 = 0$$

an equation whose roots are six times those of the former.

But this method of removing the fractions often adds considerably to the complexity of the equation; and, unless we are anxious to free the result from its leading coefficient as well as from fractions, should be employed only in favorable cases, where some *submultiple* of the common multiple will effect the object. By clearing the above equation in the ordinary way, that is, by multiplying by 6, we convert it into

$$6x^4+3x^3-2x^2-x+60=0$$

an equation much simpler in appearance than the former, and of which the roots are the same as in the original. Yet, as just intimated, there are cases in which the transformation of an equation into another whose roots shall be a certain multiple of the roots of the proposed, will lead to a simpler form than would result from clearing the proposed from fractions in the ordinary way. For instance, if the example given above had been

$$x^4+\frac{1}{2}x^3-\frac{1}{3}x^2-\frac{1}{36}x+\frac{5}{648}=0$$

we should have converted it into

$$x^4+3x^3-12x^2-x+5=0$$

an equation whose roots are six times as great, by using the multipliers 6, $6^2$, $6^3$, $6^4$; 6 being a small submultiple of 648.

If decimals, however, occur in the coefficients of an equation they should be removed in the ordinary way, or in a way that is in reality equivalent to it. We should commence with that coefficient which has the greatest number of decimal places, suppose, for instance, $p$ places, and then remove all these decimal figures to the *left* of the decimal point, or, which is the same thing, strike out the point altogether.

Then in every other coefficient the *same number*, $p$, of decimal places must, in like manner be removed to the integral side of the point; the deficiency from so many as $p$ places being supplied by ciphers. The decimal points are thus virtually struck out; ev term having been multiplied by $10^p$. The equation

$$1{\cdot}24x^3-{\cdot}7x^2+26x+16{\cdot}3=0$$

is thus converted into

$$124x^3 - 70x^2 + 2600x + 1630 = 0$$

It may be further remarked, in connexion with the foregoing problem, that if all the roots of an equation be multiplied by $-1$ the positive roots will be changed into negative, and the negative into positive; and as the successive powers of $-1$ are $-1, +1, -1, +1$, &c., it follows that to transform an equation into another whose roots shall be the same in value, but have changed signs, we must change the alternate signs of the original equation, beginning at the second term of it. This truth we have already established, from other considerations, at (24).

### PROPOSITION II. PROBLEM.

(36.) To transform an equation into another, of which the roots shall be greater or less than the roots of the original by any given quantity.

The general method of effecting this transformation will be abundantly apparent after showing the manner of proceeding in an equation of some specified degree. We shall for this purpose take the equation of the fourth degree, viz.

$$A_4x^4 + A_3x^3 + A_2x^2 + Ax + N = 0 \quad . \ . \ . \ . \ [1]$$

and shall suppose it to be required to transform this into another equation of the same degree whose roots shall each of them differ from those of the proposed equation by some given quantity $r$, either positive or negative.

It is immediately obvious that the transformed equation sought would be obtained by substituting in the proposed $x' + r$ for $x$. For the resulting equation in $x'$ would necessarily have its roots related to those of the original, by the condition $x' = x - r$; that is the several values of $x'$ would be *less* than those of $x$ by the quantity $r$, if $r$ were positive, and *greater* if $r$ were negative.

All therefore that we have to accomplish in the present problem is the discovery of some brief arithmetical process for obtaining the result of the substitution spoken of, without formally going through the several involutions of $x' + r$, multiplying the powers as they arise by the coefficients A, $A_2$, $A_3$, $A_4$, and then finally arranging the terms thus brought out.

In order to avoid this trouble let us remember that, whatever be the final result just adverted to, its *form* will be this, viz.

$$A_4 x'^4 + A'_3 x'^3 + A'_2 x'^2 + A' x' + N' = 0 \quad . \; . \; . \; . \quad [2]$$

and that from this we should return to the original equation by giving to $x'$ its original value, viz. $x - r$. Hence, substituting $x - r$ for $x'$ in the preceding equation, we infer the following identity, viz.

$$A_4 (x - r)^4 + A'_3 (x - r)^3 + A'_2 (x - r)^2 + A' (x - r) + N' =$$

$$A_4 x^4 + A_3 x^3 + A_2 x^2 + Ax + N$$

from which the unknown coefficients in [2] are obtained with ease and rapidity from the following considerations:

If we divide the first member by $x - r$ the remainder must be N'. But as the two members are identical the same remainder and the same quotient must arise whichever member we divide. The division, therefore, of the second member, that is of the original polynomial, by $x - r$ must give N' for remainder, and for quotient the polynomial

$$A_4 (x - r)^3 + A'_3 (x - r)^2 + A'_2 (x - r) + A'$$

Also, dividing this by $x - r$ we have for remainder A', and for quotient

$$A_4 (x - r)^2 + A'_3 (x - r) + A'_2$$

Again, dividing this by $x - r$, the remainder becomes $A'_2$ and t quotient

$$A_4 (x - r) + A'_3$$

And lastly, dividing this by $x-r$ we have for the final remainder $A'_3$, and for the final quotient, $A_4$.

It thus appears that the several remainders arising from these successive divisions of the original polynomial by $x-r$ supply all the coefficients of the transformed equation [2]. And as we have exhibited at (8) a very expeditious method of executing these divisions, we are thus furnished with a process for effecting the proposed transformation, which, on the score of ease and rapidity, is much superior to the common way of substituting $x'+r$ for $x$ in the original equation, and then developing and arranging the terms.

(37.) We shall give a few examples of this transformation.

EXAMPLES.

1. It is required to transform the equation

$$x^3-2x^2-23x+60=0$$

into one whose roots shall be the roots of this increased by 2.

The constant divisor in this case is $x-2$. Hence, proceeding in conformity with the preceding directions conducting the several steps as recommended at (9), the operation will stand thus:

$$\begin{array}{llll}
A_3 & A_2 & A & N \\
1 & -2 & -23 & +60 \quad (-2=r \\
 & -2 & +\ 8 & +30 \\
\hline
 & -4 & -15 & +90 \quad \therefore N'=90 \\
 & -2 & +12 & \\
\hline
 & -6 & -\ 3 & \therefore A'=-3 \\
 & -2 & & \\
\hline
 & -8 & \therefore A'_2=-8 &
\end{array}$$

Consequently the transformed equation is

$$x'^3 - 8x'^2 - 3x' + 90 = 0$$

We may arrange the work a little differently, thus:

$$\begin{array}{rrrl}
1 & -2 & -23 & 60\ (-2 \\
 & -2 & 8 & 30 \\
\cline{2-2}\cline{3-3}
 & -4 & -15 & \\
 & -2 & 12 & \\
\cline{2-2}
 & -6 & & \\
 & -2 & & \\
\cline{2-4}
\end{array}$$

Transformed equation, $x'^3 - 8x'^3 - 3x'^2 + 90 = 0$

2. Transform the equation

$$x^3 - 2x^2 - x + 2 = 0$$

into one whose roots shall be less than the roots of this by 3.

$$\begin{array}{rrrl}
1 & -2 & -1 & 2\ (3 \\
 & 3 & 3 & 6 \\
\cline{2-2}\cline{3-3}
 & 1 & 2 & \\
 & 3 & 12 & \\
\cline{2-2}
 & 4 & & \\
 & 3 & & \\
\cline{2-4}
\end{array}$$

Transformed equation, $x'^3 + 7x'^2 + 14x' + 8 = 0$

3. Transform the equation

$$x^5 + 6x^4 + x^3 - 16x^2 - 20x - 16 = 0$$

nto one whose roots shall be greater than the roots of this by 1.

$$\begin{array}{rrrrrrl}
1 & 5 & 1 & -16 & -20 & -16 & (-1 \\
 & \underline{-1} & \underline{-4} & \underline{3} & \underline{13} & 7 & \\
 & 4 & -3 & -13 & -7 & & \\
 & \underline{-1} & \underline{-3} & \underline{6} & 7 & & \\
 & 3 & -6 & -7 & & & \\
 & \underline{-1} & \underline{-2} & 8 & & & \\
 & 2 & -8 & & & & \\
 & \underline{-1} & -1 & & & & \\
 & 1 & & & & & \\
 & -1 & & & & & \\
\hline
\end{array}$$

$$x'^5 + 0x'^4 - 9x'^3 + x'^2 + 0x' - 9 = 0$$

That is, the transformed equation is

$$x'^5 - 9x'^3 + x'^2 - 9 = 0$$

4. Transform the equation

$$x^3 - 10x^2 + 31x - 30 = 0$$

into another whose roots shall be less by unity than the roots of this.

The required equation is

$$x'^3 - 7x'^2 + 14x' - 8 = 0$$

5. Transform the equation

$$x^4 + 13x^3 + 58x^2 + 111x - 27 = 0$$

into another whose roots shall be equal to the roots of this increased by 2.

The transformed equation is

$$x^4 + 5x^3 + 4x^2 + 3x - 105 = 0$$

6. Transform the equation

$$2x^4 + 8x^3 - x^2 - 8x - 20 = 0$$

into another whose roots shall be equal to the roots of this increased by 1.

The transformed equation is

$$2x^4 - 13x^2 + 10x - 19 = 0$$

7. Transform the equation

$$6x^3 - 3x^2 + 4x - 1 = 0$$

into another whose roots shall exceed the roots of this by 3.

The transformed equation is

$$6x^3 - 57x^2 + 184x - 202 = 0$$

8. Transform the equation

$$x^3 + 8x^2 + 6x - 75{\cdot}9 = 0$$

into another whose roots shall be less by 2.

The transformed equation is

$$x^3 + 14x^2 + 50x - 23{\cdot}9 = 0$$

9. Transform the resulting equation just deduced into another whose roots shall be less by ·4.

The transformed equation is

$$x^3 + 15{\cdot}2x^2 + 61{\cdot}68x - 1{\cdot}596 = 0$$

10. Transform the equation last deduced into another whose oots shall be less by ·02

The transformed equation is

$$x^3 + 15{\cdot}26x^2 + 62{\cdot}2892x - {\cdot}356312 = 0$$

he roots of which are those of the original equation, viz., of the quation,

$$x^3 + 8x^2 + 6x - 75{\cdot}9 = 0$$

diminished by 2·42

(38.) The transformations here effected are highly useful in the actual solution of equations, the roots of which are developed figure by figure by a series of steps of this kind.

The same transformations enable us also to remove the second term from an equation, a preparative operation necessary in certain methods of solution; and one which generally facilitates the process whatever mode of solution be employed. We shall make it the subject of a distinct problem.

### PROPOSITION III. PROBLEM.

(39.) To transform an equation into another in which the second term shall be absent.

It has been seen in the preceding problem that in diminishing the roots of an equation of the *n*th degree, by a given quantity *r*, this quantity was added to the second coefficient *n* times; so that when it is required that the result of these *n* additions should be zero, it is plain that *r* must be minus the *n*th part of the second coefficient. Hence, in order to effect the desired transformation we must diminish the roots of the proposed equation by minus the *n*th part of the coefficient of its second term; the coefficient of the first term being unity.

### EXAMPLES.

1. It is required to transform the equation

$$x^3 - 9x^2 + 20x - 12 = 0$$

into another in which the second term shall be absent.

Here $r = \frac{9}{3} = 3$. Hence the required equation is deduced as follows:

$$\begin{array}{rrrrl}
1 & -9 & 20 & -12 & (3 \\
 & 3 & -18 & 6 & \\
\cline{2-2}\cline{3-3}
 & -6 & 2 & & \\
 & 3 & -9 & & \\
\cline{2-2}
 & -3 & & & \\
 & 3 & & & \\
\hline
x'^3 & +\,0x'^2 & -\;7x' & -\;6 = 0 &
\end{array}$$

Transformed equation $x'^3 + 0x'^2 - 7x' - 6 = 0$

The roots of this equation are equal to those of the proposed diminished by 3; so that if the roots of the transformed equation be discovered, those of the original are immediately deducible from them.

2. Transform the equation

$$x^3 - 6x^2 + 5 = 0$$

into one which shall want its second term.

The transformed equation is

$$x'^3 - 12x' - 11 = 0$$

3. Transform the equation

$$x^4 + 4x^3 - 49x^2 + 104x - 60 = 0$$

into another whose second term shall be absent.

The transformed equation is

$$x'^4 - 55x'^3 + 210x' - 216 = 0$$

It is scarcely necessary to remark that if the coefficient of the second term in the given equation be not a multiple of the exponent $n$, the coefficients of the transformed equation will involve fractions.

# CHAPTER III.

## ON THE LIMITS TO THE REAL ROOTS OF AN EQUATION.

(40.) THE first step towards the solution of a numerical equation is the determination of the number and situations of its real roots. This is called the *analysis* of the equation. By the situation of a root is to be understood the place which it occupies in the arithmetical scale; which place is said to be determined when we have discovered between what two consecutive numbers in that scale it lies.

It may be proper, however, in order to preclude the possibility of misapprehension, to explain what we mean by the *arithmetical scale* in reference to this enquiry, and with this view we remark that the term, as here employed, means a scale of integer numbers, commencing at 0, and proceeding in both directions, ascending positively up to $\infty$, and descending negatively down to $-\infty$. In each direction the scale is divided from 0 into distinct compartments or stages, each compartment consisting alike of *ten* equal grades or steps; the grades in one compartment are not however individually equal to those in the preceding compartment, but always exceed them in a *tenfold* proportion.

From this brief description it will be perceived that the positive part of our arithmetical scale, with the compartments distinguished, must be as follows

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, | 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, |
100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, | 1000, 2000, &c. &c.

and that the negative part must be the same with the addition of the minus signs to each number.

The scale is here extended only upwards from 1 towards the right; its continuation towards the left would obviously supply the following grades, which we shall here consider as carried on in that direction, commencing at the 1 on the left, in the lower row :

. . . . . . . . . | ·001, ·002, ·003, ·004, ·005, ·006, ·007, ·008, ·009, ·01 |

·01, ·02, ·03, ·04, ·05, ·06, ·07, ·08, ·09, ·1 | ·1, ·2, ·3, ·4, ·5, ·6, ·7, ·8, ·9, 1 |

(41.) If a root of an equation should happen to be one of the numbers in this scale, the process by which the situation of such root is determined must be capable of conducting us accurately to the root itself. But if, as is much more frequently the case, the root should be some number, whether integral or decimal, not in the scale, then the object is merely to discover between what two consecutive numbers in the scale the root really lies. When this is accomplished the *situation* of the root is said to be determined; and that this determination is an important step towards the actual evolution of the root, the student will immediately perceive when he is informed that the process by which this evolution is effected is applicable only after the *leading figure* of the root has been discovered by some independent means. The discovery of the situation of a root at once suggests the leading figure of that root; for the leading figure is invariably that of the numerically smaller of the two numbers between which the root lies, whether it lie in the positive or in the negative region of the scale. For instance, if a root be found to lie between 9 and 10, the first figure of it is necessarily 9; if between 300 and 400, the first figure of it is necessarily 3, which 3 is in the *hundred's* place; if between — 2000 and — 3000, the first figure is necessarily — 2, which — 2 occupies the place of *thousands*: and so on.

It is easy to see, since the determination of the leading figure of a root and the place it occupies is all that is necessary to furnish a sufficient basis for those approximative methods by which the following figures of the root are evolved one after another :—we say it is thus easy to see why our arithmetical scale, after the leading compartment, contains first, only every tenth figure in the natural series of numbers, then only every hundreth, then only every thousandth, &c.; the filling up of the intervals would evidently only complicate the scale to no useful purpose. The determination of the *first figure* only, of the number falling between any two in the scale, being all that we require to know about it, the numbers already inserted are, as we have just seen, amply sufficient.

(42.) In the second section of this work, when we come to consider the actual analysis of equations, we shall show how to deduce, from the polynomial forming the first member of any equation, certain dependent polynomials of inferior degrees, which shall have the remarkable property of making known, by the signs which they furnish when any two numbers chosen at pleasure are separately substituted for the unknown in them, how many roots of the equation lie between these chosen numbers; so that commencing at 0, and substituting pair after pair, the numbers of our arithmetical scale, in these polynomials, we should thus be able to discover the situations of all the roots of any numerical equation. Now, in order to diminish the number of needless substitutions arising from trying pairs of numbers, between which roots cannot possibly lie, it is necessary that we be provided with the means of ascertaining beforehand *limits*, as little remote from one another as possible, within the bounds of which all the real roots of the equation must be comprised.

It is the object of this chapter to investigate such limits, confining ourselves, however, to the determination of those only which are the most readily applied in actual practice; referring for a more complete theory of this subject to the Treatise on the *Theory of Equations in general.*

### PROPOSITION I. THEOREM.

(43.) In any equation whose second term is negative and all the other terms positive, the coefficient of the second term taken positively is a *superior limit* to the positive roots; that is, it is greater than the greatest of those roots.

Let the equation be

$$x^n - A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots\ldots Ax + N = 0$$

which, since

$$x^n - A_{n-1}x^{n-1} = (x - A_{n-1})\,x^{n-1}$$

may be written as follows:

$$(x - A_{n-1})x^{n-1} + A_{n-2}x^{n-2} + \ldots\ Ax + N = 0$$

Now if $A_{n-1}$ be substituted in this equation for $x$, the first term will vanish; and all the other terms being positive the entire result of this substitution must be positive. And if a number greater than $A_{n-1}$ be substituted, not only will the terms after the first be still positive, but the first term itself, which before vanished, will also be positive. Consequently, the first member of the proposed equation can never become zero for positive values of $x$ so great as $A_{n-1}$. Hence this quantity must exceed the greatest positive root of the equation.

### PROPOSITION II. THEOREM.

(44.) In any equation the greatest negative coefficient taken positively and increased by unity is a superior limit to the positive roots.

Let the equation be

$$x^n + A_{n-1}\,x^{n-1} + \ldots\ldots Ax + N = 0 \ . \ . \ [1]$$

and suppose $A_m$ to represent the greatest negative coefficient, which let be $-q$.

Take now another equation of the same degree, but in which *all* the coefficients are negative and equal to $-q$, viz. the equation

$$x^n - qx^{n-1} - \ldots\ldots - qx - q = 0$$

or

$$x^n - q\,(x^{n-1} + \ldots\ldots + x + 1) = 0 \ . \ . \ . \ . \ [2]$$

Then it is plain that whatever substitutions for $x$ give positive results here, the same must give positive results in the proposed equation.

The terms within the parentheses in the equation [2] form a geometrical series in which the common ratio is $x$; so that we have for their sum (*Algebra*, page 98) $\frac{x^n-1}{x-1}$. And consequently the equation may be written

$$x^n - q\frac{x^n-1}{x-1} = 0 \;.\;.\;.\;.\;.\; [3]$$

Now it is plain that if $q$ be equal to $x-1$ the first member of this will be positive, viz. $+1$; and that it will be a positive quantity still greater if $q$ be less than $x-1$. But when $q=x-1$ we must have $x=q+1$; and when $q<x-1$ we must have $x>q+1$; so that the first member of [3] continues positive for all positive values of $x$ from the value $q+1$ upwards. Hence the first member of the proposed equation continues positive for all such values; that is, the greatest negative coefficient taken positively and increased by unity exceeds the greatest root of the equation.

(45.) In this and the preceding proposition superior limits to the *positive* roots only are considered. But this is all that is necessary; because by changing the alternate signs of an equation, commencing the change at the second term, we convert the positive roots into negative, and *vice versâ*; so that for an inferior limit to the negative roots of any equation we need only make this change in the signs of the terms, then find a superior limit to the positive roots, and afterwards change the sign of this limit.

(46.) It is plain that if the greatest coefficient in any equation, without regard to sign, be increased by unity, the number we thus get, when taken positively and negatively, must furnish limits between which all the real roots of the equation must lie. But these limits will obviously seldom be so close as those determined by the present proposition.

### PROPOSITION III. THEOREM.

(47.) If each negative coefficient be taken positively, and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus obtained when increased by unity will be a superior limit to the positive roots.

As already noticed in Proposition II, the known formula for the sum of a geometrical series furnishes the equation

$$\frac{x^m-1}{x-1}=x^{m-1}+x^{m-2}+x^{m-3}+\ldots\ldots x+1$$

from which we get, by multiplying by $x-1$, and then transposing the 1,

$$x^m=(x-1)x^{m-1}+(x-1)x^{m-2}+(x-1)x^{m-3}$$
$$+\ldots(x-1)x+(x-1)+1.$$

Hence we may replace the several powers of $x$ in any equation, each by an expression like this; in which expression are involved *all* the powers of $x$ inferior to that thus replaced.

Suppose the *positive* terms only of the equation

$$A_nx^n+A_{n-1}x^{n-1}+A_{n-2}x^{n-2}\ldots\ldots Ax+N=0\ \ldots\ldots\ [1]$$

to be developed in this manner. Then if the several developments are written in horizontal rows below one another, so that *like* powers of $x$ may range vertically, it is plain that the sum of all these positive terms will be of the form

$$A_n(x-1)x^{n-1}+A'_{n-1}(x-1)x^{n-2}+A'_{n-2}(x-1)x^{n-3}$$
$$+\ldots\ldots A'(x-1)+N'\ \ldots\ldots\ [2]$$

And as in this sum there occur all the powers of $x$, inferior to the leading power $x^n$ of the proposed equation, it is obvious that if we now range in their proper places under the like terms of this sum the several undeveloped negative terms of [1] which have been omitted, more than one such negative term cannot possibly have place under any proposed term of [2], if any at all, because in different negative terms the powers of $x$ will be different.

Suppose the first negative term that occurs in [1] to be $-A_p x^p$, the proper place of this under [2] will be below $A'_{p+1}(x-1)x^p$, and the sum of these two terms will always be positive, whatever positive value be given to $x$, provided only that

$$A'_{p+1}(x-1) > A_p \text{ or } x > \frac{A_p}{A'_{p+1}} + 1 \;\ldots\ldots\; [3]$$

and the sum will be zero, provided these two expressions are equal. And in either of these hypotheses all the terms of [2] preceding the term $A'_{p+1}(x-1)x^p$ will be positive; since the coefficients of them being all positive, they must remain positive so long as $x$ exceeds unity.

In like manner if the next negative term that occurs in [1] be $-A_q x^q$ the sum of it and the like term in [2] will be positive for all values of $x$, provided

$$A'_{q+1}(x-1) > A_q \text{ or } x > \frac{A_q}{A'_{q+1}} + 1 \;\ldots\ldots\; [4]$$

and the sum will be zero if $x$ be equal to, instead of greater than, this expression. And as before all the terms of [2] preceding $A'_{q+1}(x-1)A_q$, up to the term last considered, will be positive in either hypothesis. The reasoning is the same for the next negative term in [1], and so on to the last.

If of the fractions occurring in the expressions [3], [4], &c. thus obtained, we select the greatest, this will obviously satisfy *all* the conditions [3], [4], &c.; that is to say, if $x$ be made either equal to, or greater than, this fraction increased by unity, then the entire sum of [2], together with the negative terms of [1] which are absent from [2], must be positive; or which is the same thing the polynomial [1] itself must be positive.

Now from the manner in which the development [2] has been formed it is plain that the coefficient $A'_{q+1}$, of any term

$A'_{q+1}(x-1)x^q$, is the sum of all the *positive* coefficients in [1] which precede the term $A_q x^q$.

Hence if fractions be formed by taking each negative coefficient positively, and dividing it by the sum of all the positive coefficients which precede it, the greatest of these fractions increased by unity will exceed the greatest positive root of the equation.

(48.) This proposition obviously includes Proposition II as a particular case; and as a *general* theorem is one of the most convenient and effective in its practical application that can be given. There are, however, some particular rules, for determining a superior limit, when the equation has a certain specified form, which are preferable to that furnished by the present proposition, as the limit supplied by them is closer to the greatest root itself. These rules, however, on account of their limited application we shall not investigate at length. It will be sufficient to select from among them that which is most likely to come into requisition in actual practice. This we shall make the subject of the following proposition.

### PROPOSITION IV. THEOREM.

(49.) If the second term of an equation be absent, the third term negative, and all the others positive, the square root of the negative coefficient is a superior limit to the positive roots of the equation.

The equation with the specified signs is of the form

$$x^n - A_{n-2}x^{n-2} + A_{n-3}x^{n-3} + \ldots\ldots Ax + N = 0$$

which may be otherwise written

$$(x^2 - A_{n-2})\,x^{n-2} + A_{n-3}x^{n-3} + \ldots\ldots\ldots Ax + N = 0.$$

If $x = \sqrt{A_{n-2}}$ the first term vanishes, and all the rest are positive; and if $x$ be of greater value, the first term as well as all the others will be positive. Hence $\sqrt{A_{n-2}}$ exceeds the greatest positive root of the equation.

(50.) It may be remarked in reference to the propositions now established, that all the methods for ascertaining a superior limit to the positive roots of an equation suppose the existence of at least one negative coefficient.* If no such coefficient entered the equation then, as proved at (29), that equation could not possibly have any positive roots; so that those of them which are not imaginary must be negative. And a limit below the least of these may be determined by the preceding theorems after changing the alternate signs of the terms, as already hinted at (45).

(51.) As by thus simply changing $x$ into $-x$ we are enabled to discover an inferior limit to the *negative* roots, so by changing $x$ into $\frac{1}{x}$ we may determine an inferior limit to the *positive* roots; and thence, by the change before noticed, a superior limit to the negative roots; for the greatest root of the equation thus changed will obviously be the smallest root of the original, and vice versâ; the one set of roots will in fact be the reciprocals of the other set: so that if a quantity be found to exceed the greatest root in the one set, the reciprocal of that quantity must fall below the smallest root in the other set, and conversely.

This change of an equation into another whose roots shall be the reciprocals of those of the former, being effected by the simple substitution of $\frac{1}{x}$ for $x$, merely amounts to the writing of the coefficients in reverse order, without disturbing the powers of $x$. Thus putting $\frac{1}{x}$ for $x$ in the equation

$$A_n x^n + A_{n-1} x^{n-1} + \dots\dots A_2 x^2 + Ax + N = 0$$

converts it into

$$\frac{A_n}{x^n} + \frac{A_{n-1}}{x^{n-1}} + \dots\dots \frac{A_2}{x^2} + \frac{A}{x} + N = 0$$

---

* It may be proper to observe here that in all that has preceded, the absolute quantity N is comprehended under the general name "coefficient," being indeed the coefficient of $x^0$.

which upon clearing fractions becomes

$$A_n + A_{n-1}x + \ldots\ldots A_2 x^{n-2} + Ax^{n-1} + Nx^n = 0$$

or rather

$$Nx^n + Ax^{n-1} + A_2 x^{n-2} + \ldots\ldots A_{n-1}x + A_n = 0$$

which differs from the proposed only in the reversion of the coefficients.

We thus see that the theory of the limits of the roots of an equation, comprehending the inferior limits as well as the superior, and the negative roots as well as the positive, becomes complete so soon as all that concerns the *superior* limits of the *positive* roots is established; since, by one or other of the easy preparatory changes described above, this single case is made virtually to comprehend all the others.

(52.) Sometimes, however, a close superior limit to the positive roots of an equation may be readily discovered, without the aid of the foregoing theorems, simple as their application is, by a few moments' contemplation of the equation itself, for the purpose of roughly estimating the comparative importance of the positive and negative terms.

In this tentative method it will be advisable to bring all the positive terms together, and all the negative terms together, separating the former aggregate from the latter by the sign of inequality $>$, implying the condition which a positive limit must always fulfil, and which must also be fulfilled for every value of $x$ above any such limit.

For example: let the equation

$$x^3 - 6x^2 + 9x - 1 = 0$$

be proposed

Then we must fulfil the inequality

$$x^3 + 9x > 6x^2 + 1$$

or, dividing by the highest power of $x$, the inequality

$$1+\frac{9}{x^2} > \frac{6}{x}+\frac{1}{x^3}$$

and which we see is fulfilled by $x=4$; and obviously by every higher number.

Hence 4 is a superior limit to the positive roots; and it is a smaller number, that is a closer limit, than either of the preceding propositions would furnish. Propositions II and III, the only ones applicable to this example, each give 7 for a superior limit.

Again let the equation

$$x^4+x^3-15x^2-19x-3=0$$

be proposed.

Then we must have

$$x^4+x^3 > 15x^2+19x+3$$

or

$$1+\frac{1}{x} > \frac{15}{x^2}+\frac{19}{x^3}+\frac{3}{x^4}$$

which is obviously satisfied for $x=5$, and for every higher number. By proposition III the limit would be double this. That proposition, however, is not to be undervalued on this account; it furnishes, perhaps, upon the whole, the best and most expeditious direct method of finding a limit that can be proposed; and will in many cases be obviously preferable to the tentative mode of proceeding just exemplified. For instance, we should never think of substituting this latter mode for the direct method in such an equation as this:

$$3x^4+9x^3-4x^2+23x-39+72=0$$

which by proposition III is at once seen to have

$$\frac{39}{3+9+23}+1$$

for a superior limit; and therefore the integer 3 must also b superior limit.

The superior integral limit of the positive roots in each of the following equations is determined by this rule.

1. $x^5+5x^3-10x^2+5x-1=0$ Limit $=3$
2. $5x^4+12x^3+6x^2-6x-2=0$ 2
3. $4x^7-6x^6-7x^5+8x^4+7x^3-23x^2-22x-5=0$ 3

### PROPOSITION V. THEOREM.

(53.) If the real roots of an equation be ranged in the order of their magnitude, commencing with the greatest positive root, or the root nearest to $+\infty$, and terminating with the least negative root, or the root nearest to $-\infty$, thus furnishing the descending series

$$a,\ a_2,\ a_3,\ a_4,\ \ldots\ldots$$

and if a number $b$, exceeding the greatest root $a$, be substituted for $x$ in the polynomial forming the first member of the equation, the result will be *positive;* if a number $b_2$, in magnitude between $a$ and $a_2$, be substituted the result will be *negative;* if a number $b_3$, between $a_2$ and $a_3$, be substituted the result will be *positive,* and so on.

The polynomial in question is formed from the *simple* factors

$$(x-a)(x-a_2)(x-a_3)(x-a_4)\ \ldots\ldots$$

multiplied by the factor of an even degree involving the imaginary roots (21).

Omitting this latter factor, should such exist, let us examine the effect of the proposed substitutions upon the product of the real factors. Putting then $b$ for $x$ in each we have

$$(b-a)\,(b-a_2)\,(b-a_3)\,(b-a_4)\ \ldots\ldots\ \text{a } \textit{positive} \text{ number,}$$

ause, by hypothesis, all these factors are positive.

Putting $b_2$ for $x$ we have

$(b_2 - a)\ (b_2 - a_2)\ (b_2 - a_3)\ (b_2 - a_4)$ . . . . ., a *negative* number,

because the first factor is negative, and all the others positive.

Putting $b_3$ for $x$ we have

$(b_3 - a)\ (b_3 - a_2)\ (b_3 - a_3)\ (b_3 - a_4)$ . . . . ., a *positive* number,

because the first two factors are negative and all the others positive. And continuing in this manner to substitute for $x$ a series of numbers each of which occupies a place between two successive roots, it is plain that the results of the substitutions will continue to be alternately positive and negative, till the last of the roots is passed over, that is till our substitution has become so small as to be less than the least root of the equation. And we should thus be furnished with as many changes of sign as there are real roots in the equation; provided, that is, that these roots differ in value from one another.

(54.) It follows from this that if *two* roots, or indeed any even number, lie between two numbers substituted for $x$, our results must have the same sign; like as would happen if no roots were comprehended between the numbers substituted; so that when the same signs result from any pair of substitutions all that we can infer is that an odd number of roots cannot lie between those numbers; but without additional information we can affirm nothing as to whether pairs of roots exist in the interval or no roots at all.

When pairs of equal roots exist in an equation it is plain that *they* cannot be separated. For whatever two numbers comprehend one must comprehend a pair. If an odd number of equal roots enter, these may be separated from the rest, but not from one another. And in all cases the difficulty of separating the roo increases with their proximity to one another.

We thus see that the attempt to analyse an equation, or t determine the number and situations of its real roots, by th tentative operations suggested by this proposition, would often be tedious, and in many cases but partially successful; for when like signs were furnished by any two trial-numbers we might be at a loss to determine whether this permanence of sign arose from the entire absence of roots in the interval, from the existence of equal roots, or from the entrance of unequal roots differing but little from each other. Nevertheless till very lately it was only by this vague and uncertain kind of guess-work,—controlled indeed by the truths established respecting the *limits* of the roots, and by the rule of signs—that we could hope to discover the number and places of the real roots of an equation. Now, however, as remarked at (42), we are in possession of a method, to be explained in next section, by which we can infallibly determine not only whether or not roots lie between any proposed numbers, but also their exact number whenever they exist. The following proposition, with which we shall terminate the present section on the theoretical properties of equations, has an immediate bearing upon the method alluded to; and, with the observations subjoined to it, should be clearly understood by the student.

(55.) Before entering upon it, however, it may be proper to explain that if an equation be found of which the real roots lie between those of any given equation, the former is called a *limiting equation* to the latter. There is, of course, an endless variety of such limiting equations to any given equation; and, knowing the real roots of the given equation, limiting equations might be framed in any number by assuming roots anywhere between the given roots, and constructing equations involving these assumed roots (12). These new equations would be in degree a unit at least below the proposed, and would all be limiting equations to it. But the great object is to discover an equation that shall be a limiting equation to one already given without knowing the roots of either. And this is the object of the proposition now to be investigated.

### PROPOSITION VI. PROBLEM.

(56.) An equation being given to determine another of a degree immediately inferior, such that the real roots of the latter may lie between the real roots of the former.

We shall suppose the degree of the given equation not to exceed the *fourth*, as the analysis of equations of more advanced degrees will not be entered upon in the present volume, and the problem before us is merely preparative to that analysis. From the mode of reasoning pursued, the student will perceive that the conclusion to which we shall be conducted may be extended to equations of the higher orders.

Let then the proposed equation be

$$A_4x^4 + A_3x^3 + A_2x^2 + Ax + N = 0 \; . \; . \; . \; . \; [1]$$

where $A_4$ will be 0 if the equation is a cubic.

Arrange the *real* roots as in last proposition, commencing with the greatest and descending to the least:

$$a, \quad a_2, \quad a_3, \quad a_4$$

And suppose for the present that all the roots are real.

Then it is plain that if these roots be diminished by $r$, the roots of the transformed equation will be

$$a - r, \quad a_2 - r, \quad a_3 - r, \quad a_4 - r \; . \; . \; . \; . \; [2]$$

the form of this equation being

$$A_4x'^4 + A'_3x'^3 + A'_2x'^2 + A'x' + N' = 0 \; . \; . \; . \; . \; [3]$$

And if the coefficients here exhibited be all divided by the first coefficient $A_4$, the coefficient which $x'^3$ will then take will be sum of the roots [2] with their signs changed, the coefficient $x'^2$ the sum of their products two and two, the coefficient of that is, $\frac{A'}{A_4}$, the sum of their products, three and three, signs being changed. Hence for A′ we have the expression

$$\begin{aligned} A' = A_4\{&(r-a)(r-a_2)(r-a_3)+ \\ &(r-a)(r-a_2)(r-a_4)+ \\ &(r-a)(r-a_3)(r-a_4)+ \\ &(r-a_2)(r-a_3)(r-a_4)\} \quad \ldots\ldots [4] \end{aligned}$$

But A′, the coefficient of the last term but one in the equation which arises from diminishing the roots of [1] by $r$, is very easily obtained by incorporating $r$ with the coefficients of [1], as explained at (37); the operation being as follows, viz.

$$\begin{array}{lllllll} A_4 & + & A_3 & + & A_2 & + & A \quad (r \\ & & A_4r & & A_4r^2 + A_3r & & A_4r^3 + A_3r^2 + A_2r \\ \hline & & (A_4r + A_3) & + & (A_4r^2 + A_3r + A_2) & + & (A_4r^3 + A_3r^2 + A_2r + A) \\ & & A_4r & & 2A_4r^2 + A_3r & & 3A_4r^3 + 2A_3r^2 + A_2r \\ \hline & & (2A_4r + A_3) & + & (3A_4r^2 + 2A_3r + A_2) & + & (4A_4r^3 + 3A_3r^2 + 2A_2r + A) \end{array}$$

Consequently

$$A' = 4A_4r^3 + 3A_3r^2 + 2A_2r + A \quad \ldots\ldots [5]$$

Now, by aid of the last proposition, it is easily seen that if this expression be equated to zero we shall have an equation whose real roots lie between those of the proposed.

For as [4] and [5] are identical, put $a$ for $r$ in [4] instead of in [5]: the result is

$$(a-a_2)(a-a_3)(a-a_4),\ \text{a } \textit{positive} \text{ quantity,}$$

ause all the factors are positive, $a$ being by hypothesis the atest root of [1].

Put now $a_2$ for $r$; the result is

$$(a_2 - a)(a_2 - a_3)(a_2 - a_4), \text{ a } \textit{negative} \text{ quantity,}$$

because the first factor is negative and the others positive.

In like manner, put $a_3$ for $r$; the result is

$$(a_3 - a)(a_3 - a_2)(a_3 - a_4), \text{ a } \textit{positive} \text{ quantity,}$$

because the first two factors are negative and the other positive.

Lastly, put $a_4$ for $r$: the result is

$$(a_4 - a)(a_4 - a_2)(a_4 - a_3), \text{ a } \textit{negative} \text{ quantity,}$$

since the factors are all negative.

Hence the equation $A' = 0$, that is,

$$4A_4r^3 + 3A_3r^2 + 2A_2r + A = 0 \quad . \; . \; . \; . \quad [6]$$

is a *limiting equation* to the proposed, since the roots of this lie between those of [1], being situated with respect to those of [1] as follows:

$$\begin{array}{ccccccc} a & & a_2 & & a_3 & & a_4 \\ & r & & r_2 & & r_3 & \end{array}$$

(57.) We have supposed the roots to be all real; but if two of them, as $a_3$, $a_4$, were imaginary, similar conclusions would follow; that is, the substitutions of the real roots $a$, $a_2$, for $r$ in the equation $A' = 0$ would give, as above, first a *positive* result and then a *negative* result; because the factors $(r - a_3)$, $(r - a_4)$ furnish a quadratic product which continues *plus* whatever value be put for $r$ (21). Thus, as before, the equation $A' = 0$ has necessarily a real root between $a$ and $a_2$; but in this case *all* the roots of $A' = 0$ may, for aught we know to the contrary, lie between the same limits. As but one change of sign in the results of $A'$ is possible in the case supposed, we know that an *odd* number of the roots of $A' = 0$ must lie between the numbers to which this change is due; so that either all three of the roots are situated between $a$ and $a_2$, or else only one: the other two may indeed be imaginary.

When the roots of the proposed are all real, then, as shown above, the roots of the limiting equation are real too; and are so distributed among those of the proposed that each occupies a distinct interval between two. But when imaginary roots enter the proposed equation, all that the foregoing reasoning authorizes us to conclude is that the intervals between the real roots will be occupied each by an odd number of the real roots of the limiting equation; so that generalizing the preceding results we may say that if an equation have $p$ real roots the limiting equation, determined as above, must have at least $p-1$ real roots; and cannot therefore have more imaginary roots than the original has, although it may have fewer.

(58.) The limiting equation [6] just determined, if written with $x$ in place of $r$, is

$$4A_4x^3 + 3A_3x^2 + 2A_2x + A = 0 \quad . \; . \; . \; . \quad [7]$$

the first member of which is related in a very remarkable manner to the first member of the original equation

$$A_4x^4 + A_3x^3 + A_2x^2 + Ax + N = 0 \quad . \; . \; . \quad [8]$$

for we see that the coefficients of the former are all obtained from those of the latter by simply prefixing to each of these the *exponent* of that power of $x$ to which it belongs.

Thus the first coefficient of [7] is only the first of [8] with the exponent 4 prefixed; the second coefficient of [7] is the second of [8] with the exponent 3 prefixed, and so on.

It is also worthy of being borne in remembrance, as the truth will be referred to hereafter, that the first member of the limiting equation, as for instance the first member of [4] when $x$ is put for $r$, is formed from the simple factors of the original polynomial, by multiplying every $n-1$ of these factors together ($n$ being the degree of the equation) and adding the results.

our equation had been a cubic instead of a biquadratic then
A above would have been zero, and the same relation would of
c se have had place; as also if $A_4$ and $A_3$ had both been zero, or
t equation a quadratic. And by generalizing the reasoning we

should have found the relation to have been invariable, whatever the degree of the original equation had been *.

On account of this relation the limiting equation may always be at once written down from an inspection of the proposed. The rule being as follows:

Multiply every term in the proposed equation by the exponent of $x$ in that term, and then diminish each exponent by unity: the limiting equation will thus be exhibited.

For example, if the equation be

$$5x^4 - 4x^3 + 2x^2 + 3x - 12 = 0$$

the limiting equation will be

$$20x^3 - 12x^2 + 4x + 3 = 0$$

And if the proposed equation be

$$20x^3 - 24x^2 + 4x + 3 = 0$$

the limiting equation will be

$$60x^2 - 48x + 4 = 0$$

Also the limiting equation to this latter will be

$$120x - 48 = 0$$

We may infer, therefore, that the root of this last equation will lie between the two real roots of the preceding quadratic; that these latter will lie between those of the preceding cubic, and lastly that the real roots of the cubic will lie between those of the biquadratic.

(59.) Although, as already remarked, every equation wl e roots lie between those of another equation is called a limit g

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* See the *General Theory of Equations*.

equation to that other, yet a limiting equation determined as above is called emphatically *the* limiting equation. If the roots of it could be determined as readily as the equation itself can be formed, we should be at once in possession of a sure method of discovering the places of all the real roots of any equation. For as the roots of the limiting equation lie, as we have just seen, an odd number of them, between every two consecutive roots of the original equation, it is plain that more than one root of the original can never lie between consecutive roots of the limiting equation; so that calling these latter roots $r$, $r_2$, $r_3$, &c., it is plain that if we substitute for $x$, in the original polynomial, the successive terms of the series

$$\infty, r, r_2, r_3, , \ldots\ldots, -\infty \ldots. [9]$$

the changes of sign which result will mark the exact number of real roots and will point out where they lie.

This is an interesting and at the same time a useful truth; but it would not be generally available for the purpose of analysing an equation even though the roots $r$, $r_2$, $r_3$, were known; because the substitution of these roots, when they are incommensurable, would be too laborious an operation to be of any practical value.

(60.) But that the deduction is not without its use will be seen by showing its applicability to the determination of the nature of the roots of an equation, especially of a cubic, from the character of the coefficients. Let the cubic, deprived of its second term (39), be of the form

$$x^3+px\pm q=0$$

the coefficient of $x$ being plus, and the absolute term being either plus or minus.

The limiting equation to this is

$$3x^2+p \therefore x=\sqrt{-\frac{p}{3}}$$

this limiting equation has no real root, it is impossible that proposed can have a pair of real roots from the preceding investigation. Hence every cubic equation whose second term is absent, and the third term *plus*, must have two imaginary roots; a truth which is also deducible from the rule of Descartes, since the signs are *like* on each side of the absent term (29).

Again, let the equation be of the form

$$x^3 - px \pm q = 0$$

then the limiting equation is

$$3x^2 - p \;\therefore\; x = \pm\sqrt{\frac{p}{3}}$$

Hence, by the principle established above, if the terms of the series

$$\infty,\ \sqrt{\frac{p}{3}},\ -\sqrt{\frac{p}{3}},\ -\infty$$

be successively put for $x$ in the original equation, the number of changes in the resulting signs will express the number of real roots in that equation. The quantities actually given by these substitutions are

$$\infty,\ -2\left(\frac{p}{3}\right)^{\frac{3}{2}} \pm q,\ 2\left(\frac{p}{3}\right)^{\frac{3}{2}} \pm q,\ -\infty$$

and the signs of the two middle terms will depend upon the relation between $p$ and $q$. Thus let

$$2\left(\frac{p}{3}\right)^{\frac{3}{2}} > q \text{ or } \left(\frac{p}{3}\right)^3 > \left(\frac{q}{2}\right)^2$$

then, whether $q$ be positive or negative, we shall have the series of signs

$$+,\ -,\ +,\ -$$

exhibiting three changes, and consequently implying the existen of three real roots.

But if

$$2\left(\frac{p}{3}\right)^{\frac{3}{2}} < q \text{ or } \left(\frac{p}{3}\right)^3 < \left(\frac{q}{2}\right)^2$$

then, when $q$ is positive, we shall have the series

$$+,\ +,\ +;\ -$$

and when it is negative, the series

$$+,\ -,\ -,\ -$$

exhibiting in either case but one change, and consequently indicating the existence of only one real root.

Hence we infer, whatever be the sign of the absolute term $q$, that whenever we have the condition

$$\left(\frac{p}{3}\right)^3 > \left(\frac{q}{2}\right)^2 \quad \ldots\ldots [1]$$

the roots of the cubic are all real and unequal; but when the condition is

$$\left(\frac{p}{3}\right)^3 < \left(\frac{q}{2}\right)^2 \quad \ldots [2]$$

two of the roots are imaginary.

As the only other possible condition is

$$\left(\frac{p}{3}\right)^3 = \left(\frac{q}{2}\right)^2 \quad \ldots\ldots [3]$$

we might infer that when this exists, as the roots can then be neither unequal nor imaginary, that two at least must be *equal.* A it will be seen in next article that all three cannot possibly b qual.

.) By attending to the general investigation it will be readily perceived that if two roots $a$, $a_2$, of the proposed were equal, the root $r$, which is always, in the absence of this equality, intermediate to them in value, would have the *same* value; and that if these roots $a$, $a_2$, $a_3$, were equal, $r$, $r_2$ would also be of the same values; or if $a$, $a_2$ were one equal pair, and $a_3$, $a_4$ another, $r$ would be equal to each of the first pair, and $r_3$ to each of the other pair.

In cases of this kind the substitution of the successive terms of the series [9] would lead to the detection, not merely of the places of the equal roots, but of the roots themselves; for the original polynomial would actually vanish when those particular values of $r$ were substituted, which mark the place of equal roots.

In the preceding cubic for instance, when the condition [3] subsists, the substitution of $r=\sqrt{\frac{p}{3}}$ if $q$ is plus, or of $r=-\sqrt{\frac{p}{3}}$, if $q$ is minus, in the proposed polynomial, causes it to vanish; which shows, as inferred above, that the cubic has a pair of equal roots; the value of each being either $x=\sqrt{\frac{p}{3}}$, or $x=-\sqrt{\frac{p}{3}}$, according as $q$ is plus or minus.

There is an apparent paradox in connexion with this subject of equal roots, which it may be proper to point out and explain:

It is evidently possible for a cubic equation, as well as any other, to have *all* its roots equal: but the limiting quadratic above, viz. $3x^2-p=0$, can never have equal roots; and consequently it is impossible, by the foregoing theory, that the three roots of the equation from which this is derived can be all equal. This appears contradictory. The correct conclusion, however, from this apparent discrepancy is this, viz., that it is impossible for the equation whence the above quadratic is derived to have all its roots equal. We are warranted in affirming that this is the correct conclusion, because any other would contradict pre-established truths. But the impossibility of a cubic existing under the assumed form, when all its roots are equal, will be seen by reverting to the process by which the second term is removed fro a complete equation. As this second term is the sum of the ro s, signs being changed, we should, in the case supposed, have o diminish each of the roots of the equation by one of them. T s would exhaust them all; and thus give rise to the form $x^3 = 0$, and not to the form assumed above.

But the same conclusion may be obtained independently of any reference to external truths, by confining our attention strictly within the limits of the preceding investigation. For taking that investigation in all its generality, there *is* a case in which the two roots of $3x^2 - p = 0$ become equal, viz., the case in which $p = 0$; and in this case the condition [3] requires also that $q = 0$; that is, as concluded above, the proposed form would degenerate into $x^3 = 0$.

(62.) We remark, finally, that it is further deducible from what has already been said, that in the case of equal roots existing in an equation, the first side of that equation and the first side of the limiting equation must have a common measure, or a common factor; and that if this common factor be found, and the first member of the proposed divided by it, the quotient equated to zero will be free from equal roots, and will contain *singly* all the roots of the proposed equation. But to prove this in a manner perfectly general, for all equations, and for every hypothesis as to the multiplicity of the roots, would be out of place here. The enquiring student is therefore referred to the author's General Treatise on the Theory of Equations for a more complete discussion of this, as well as of the other parts of the theory delivered in the present section; although what is here laid down is amply sufficient for all the practical demands of the following treatise.

## SECTION II.

### *On the Analysis of Cubic and Biquadratic Equations.*

(63.) We have now come to the more practical part of our subject: the part in which we have to consider two problems, which, taken in all their generality, are the most important in algebraical science. The problems are

1. To determine the number and situations of the real roots of a numerical equation, and thence to assign the leading figure of each: and

2. Setting out with the leading figure, thus determined, to evolve the remaining figures one after another, till the complete root is obtained whenever it is commensurable, or till any assigned degree of approximation is reached whenever it is incommensurable.

The first of these general problems has been satisfactorily solved by Sturm; the second by Horner. The processes of both admit of adaptations to particular classes of equations which simplify and expedite the numerical operations. This is more especially the case with respect to the classes of equations to which this and the following section will be devoted. We shall endeavour therefore, in the discussion of these classes of equations, to present the practical operations of Sturm and Horner in this changed and improved form; and shall also attempt to give a more simple and elementary character to the theoretical investigations whence they are deduced.

# CHAPTER I.

## ON THE ANALYSIS OF EQUATIONS BY THE THEOREM OF STURM.

(64.) As just remarked, the analysis of numerical equations of all orders may be completely accomplished by a general rule recently discovered by Sturm, and known by the name of "Sturm's Theorem." The investigation of this theorem involves in it the proof of four subsidiary propositions; the first three of which, those that embody the chief peculiarities of Sturm's method, refer to certain simple and invariable relations which are found to exist among the series of remainders which successively arise in the well-known operation of seeking the greatest common measure of two algebraic expressions.

The two expressions which enter into consideration in Sturm's theorem are the polynomial which forms the first member of the proposed equation, and that which forms the first member of the limiting equation; and which, being immediately deducible from the former, as already shown (58), is called the *derived polynomial.*

Let X be put, for brevity, for the original polynomial; and $X_1$ for the derived polynomial. Then representing, in like manner, the successive quotients and remainders by single symbols, the operation to which we have alluded will stand thus:

$$\begin{array}{l} X_1)\ X\ (Q_1 \\ \quad \underline{X_1 Q_1} \\ \quad X_2)\ X_1\ (Q_2 \\ \qquad\quad \underline{X_2 Q_2} \\ \qquad\quad X_3)\ X_2\ (Q_3 \\ \qquad\quad \text{\&c. \&c. \&c.} \end{array}$$

This operation will terminate in a remainder, as $X_m$, becoming zero, provided X and $X_1$ really have a common measure; that is, provided the proposed equation $X=0$ has equal roots (62); otherwise the final remainder $X_m$ will be some *number*. It is plain that $x$ cannot enter into the final remainder; because till all the terms containing $x$ become exhausted the operation does not cease. In the propositions that follow, the roots of the proposed equation $X=0$ are supposed to be *unequal;* and consequently that, in seeking as above the common measure of X and $X_1$, we at length arrive at a remainder $X_m$, independent of $x$; that is, at some determinate *number*.

(65.) In carrying on the operation, a type of which we have just exhibited, Sturm supposes that we change all the *signs* of the several remainders as soon as they arise; altering those that are *plus* into *minus*, and those that are *minus* into *plus*. These changes cannot, obviously, have any effect upon the *absolute values* involved in the remainders; they are merely made, as matter of convenience, to subserve a purpose that might be effected by leaving the remainders with their own signs till the close of the operation for the common measure, and then introducing certain changes afterwards.

We shall regard $X_2$ then, in the foregoing process, to be the first remainder, *with changed signs*; $X_3$ the second remainder, *with changed signs;* and so on. And we shall then have the following relations, viz.:

$$\begin{aligned}
X &= X_1 Q_1 - X_2 \\
X_1 &= X_2 Q_2 - X_3 \\
X_2 &= X_3 Q_3 - X_4 \\
&\vdots \\
X_{m-2} &= X_{m-1} Q_{m-1} - X_m
\end{aligned}$$

It is from these simple and obvious relations, in conjunction with those known to exist between the roots of $X = 0$ and those of $X_1 = 0$, as to the relative situations of the two sets,—we say it is from these simple relations that Sturm has deduced one of the most important practical results that theoretical algebra has ever supplied. For he has proved that if we take the series of expressions

$$X, \ X_1, \ X_2, \ X_3 \ldots . \ X_m$$

determined as above, and substitute in them first any number $p$ for $x$, then any other number $q$, and count the number of *variations of sign* in each series of results:—

The difference between the number of variations in the first series, and the number in the second, will express exactly the number of real roots of the proposed equation $X = 0$, which are comprised between $p$ and $q$.

Hence, by substituting for $p$ and $q$, $+\infty$ and $-\infty$, or which is sufficient, by simply observing the signs of the *leading terms* in the foregoing expressions (p. 20), we may discover at once the entire number of real roots contained in the equation, since they are all necessarily comprised between $+\infty$ and $-\infty$. Moreover, the substitution of 0 and $+\infty$ will make known how many of these roots are positive and consequently how many are negative; after which the complete analysis of the equation may be easily effected, as we shall presently see.

We shall now proceed from an examination of the relations adverted to above, to establish the steps which terminate in the theorem just enunciated.

### PROPOSITION I. THEOREM.

(66.) Whatever number be put for $x$ in the series of expressions

$$X, \ X_1, \ X_2, \ X_3, \ldots . \ X_m$$

it can never happen that two consecutive expressions vanish at once.

Let

$$X_{p-1},\ X_p,\ X_{p+1}$$

be any three consecutive terms of the foregoing series.

These, as already seen, are connected together by the relation

$$X_{p-1} = X_p Q_p - X_{p+1}$$

so that if it were possible for the two conditions

$$X_{p-1} = 0,\ X_p = 0$$

to exist at the same time, it would necessarily follow that

$$X_{p+1} = 0$$

And as by the fundamental relation before quoted, we have

$$X_p = X_{p+1} Q_{p+1} - X_{p+2}$$

it would further follow that

$$X_{p+2} = 0$$

and so on till at length we should arrive at the conclusion that

$$X_m = 0$$

that is, that the last remainder is zero, which is impossible, since by hypothesis the equation $X = 0$ has no equal roots, and consequently $X$ and $X_1$ have no common measure.

Hence two consecutive expressions cannot vanish together.

### PROPOSITION II. THEOREM.

(67.) If any number substituted for $x$ in the series

$$X, X_1, X_2, X_3 \ldots X_m$$

causes one of the expressions to vanish, the two adjacent expressions, between which this one is placed, must have contrary signs.

Let the expression $X_p$ become zero when some number is substituted in it for $x$; and suppose the same number also substituted for $x$ in the adjacent expressions

$$X_{p-1}, X_{p+1}$$

The relation between the three is

$$X_{p-1} = X_p Q_p - X_{p+1}$$

which, when $X_p$ vanishes, becomes

$$X_{p-1} = -X_{p+1}$$

which shows that the adjacent expressions

$$X_{p-1}, X_{p+1},$$

for the value of $x$, which destroys the expression between them, become *equal* in value but *opposite* in sign. *Their* evanescence, along with that of $X_p$, is forbidden by last proposition.

### PROPOSITION III. THEOREM.

(68.) If the original polynomial $X$ does not vanish for either of two numbers $p$ and $r$, nor yet for any number $q$ lying between them, then however many of the subsequent polynomials in the complete series may vanish for any of these substitutions, yet no such substitution can make any change in the entire number of variations furnished by the series of polynomials; that is, if the variations be counted when $p$ is put for $x$, and when $r$ is put for $x$, and when any intermediate number $q$ is put for $x$, we shall always find them to amount to the same invariable number.

We know, in substituting a series of numbers for $x$ in any polynomial, that we continually get a recurrence of the same sign so long as we pass over no value of $x$ which renders that polynomial zero; and that the passage through such a value, that is, over a root of the equation arising from putting the polynomial equal to zero, immediately produces a change of sign.

In order, therefore, to prove the present theorem, it will be necessary to show that although at such passages as these intermediate polynomials between $X_1$ and $X_m$ necessarily change sign, yet that these changes have no effect upon the total number of variations, but that this number remains the same after the passage as before, and the same indeed at the passage itself.

Let the intermediate polynomial $X_p$ be one of those which becomes zero for $x=q$; for which value it must be remembered X does *not* vanish.* Then since for this same value of $x$ the adjacent polynomials

$$X_{p-1},\ X_{p+1}$$

give results with contrary signs (67) it follows that the three consecutive polynomials

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* As X does not vanish with $X_p$, for $x=q$, but only for numbers different from $q$, it follows that however small any of these differences may be, a difference $d$ still smaller may be assigned; so that X shall not vanish for any substitution from that of $x=q-d$, though all intermediate values of $x$, up to $x=q+d$. This may be expressed by saying that X continues unchanged in sign while $X_p$ passes through zero, from the preceding state up to the subsequent state; or that the sign of X immediately before, at, and immediately after, the passage of $X_p$ through zero, continues unchanged.

$$X_{p-1},\ X_p,\ X_{p+1}$$

must furnish one or other of these combinations of signs, viz.

$$+ \quad 0 \quad -$$

$$- \quad 0 \quad +$$

Now whether the passage of $X_p$ from the preceding state, through this zero, to the subsequent state, be a passage from + to —, or from — to +, we see at once that that passage produces no effect upon the entire number of changes of sign; for in either state there is *one permanency and one variation;* and this one variation is preserved even at the passage itself; so that, overlooking the blank occasioned by the evanescence of $X_p$, the number of *variations* in the entire series of polynomials remain undisturbed by the change in $X_p$.

We are forbidden by Proposition II. to suppose that either of the polynomials adjacent to $X_p$ can become zero with $X_p$; so that if either of these vanish at any other time, then making that the middle of three consecutive polynomials as above, we should conclude, as here, that the evanescence disturbs not the number of variations. A polynomial more distant from $X_p$ on either side may, however, vanish for the same value as $X_p$ vanishes for; but as before, it must vanish between adjacent results with *opposite signs*, and must thus leave the variations undisturbed as to number.

It follows, therefore, that if $p$ and $r$, substituted for $x$, produce no variation of sign in the original function X, nor any number lying between $p$ and $r$, the number of variations in the entire series of polynomials, will be the same when $r$ is put in them all for $x$, as when $p$ is put for $x$.

We thus learn the important fact that whatever changes take place in the *number* of the variations by the substitution of different values for $x$ in the series of polynomials, those changes can rise only from passing over roots of the proposed equation; that from passing over values which render X zero.

### PROPOSITION IV. THEOREM.

*(Sturm's Theorem.)*

(69.) If any two numbers $p$ and $q$ be substituted for $x$ in the series of polynomials

$$X, \; X_1, \; X_2, \dots X_m$$

$p$ being less than $q$, the signs of the results will present exactly as many variations *more* for $x = q$ than for $x = p$, as there are roots of the equation between $p$ and $q$.

First, let the roots of the proposed equation $X = 0$ be all real; and let them be represented in the decreasing order of their magnitude, by $a, a_2, a_3, a_4,$ &c.; and let those of the limiting equation $X_1 = 0$ be represented by $b, b_2\, b_3$ &c. Then the relative situations of these sets of roots are as follow (56)

$$\begin{array}{ccccccccc} +\infty & & a & a_2 & a_3 & a_4 & \dots\dots & & -\infty \\ & & b & b_2 & b_3 & \dots & & & \end{array}$$

Now, in substituting the same numbers for $x$ both in X, and $X_1$, it is plain that if we commence these substitutions with $x = -\infty$, and proceed continuously up to $x = +\infty$, the roots .... $a_4, a_3,$ &c. of $X = 0$ must severally be passed over before the roots of $X_1 = 0$. But as the leading term of the polynomial X has always the same sign as that of the derived polynomial $X_1$, whilst the power of $x$ in the former leading term is always a unit higher than in the latter, it follows that, proceeding from $x = -\infty$, the results of X and $X_1$ must have *opposite signs* till a root of $X = 0$ is reached. At the passage of this root the sign of X changes; and consequently, as a root of $X_1 = 0$ is not even yet reached, and therefore no change of sign been produced on $X_1$, the signs of X and $X_1$, before unlike, become *like* immediately after the passage of a root of $X=0$.

Proceeding continuously with our substitutions, a root of $X_1=0$ is at length reached, and passed, $X_1$ at the same time changing sign. The signs of X and $X_1$ are therefore again unlike; and continue so till the next root of $X=0$ is passed, when they are once more alike; and so on till all the roots of $X=0$ have been passed.

We infer, therefore, that immediately *before* a root of $X=0$ is passed there is necessarily a *variation* of sign between X and $X_1$; and that immediately *after* the passage there is necessarily a *permanency.* Thus these leading variations are successively lost and recovered; they are lost in passing the roots of $X=0$, and recovered in passing the roots of $X_1=0$.

But taking the polynomials X, $X_1$ in connexion with those which follow, that is considering the entire series

$$X, \ X_1, \ X_2, \ X_3 \ldots\ldots X_m$$

it was proved in last proposition that every one of these losses of variation must be a *permanent* loss; for although, as seen above, the variation lost between X and $X_1$ when X vanishes is recovered when $X_1$ vanishes; yet viewing the *whole row of signs* this recovery is only apparent, since the vanishing of any polynomial between X and $X_m$ has no effect on the *total number* of variations. A variation once lost can never be recovered, therefore, however the place where that loss occurs may be shifted and advanced along the row, by the vanishing of the intermediate functions.

As every loss is thus a permanent loss, and as in proceeding continuously from $p$ to $q$ there are as many such losses as roots within these limits, it follows that the signs due to $x=q$ must exhibit fewer variations than those due to $x=p$ by the number which expresses the number of roots between $p$ and $q$.

(70.) It is now easy to see that this important theorem depends essentially upon that remarkable property of our series of polynomials which is demonstrated in Proposition III, viz. that the passage of any one of them between the first and last, through zero, though necessarily producing in that one a change of sign, nevertheless leaves the entire number of variations of sign undisturbed. It is from this property that we are authorized to affirm that however often $X_1$ passes through zero, by the progressive changes in the value of $x$, that is, however often $X_1$ changes its sign, the state of the signs immediately *after* every such change is exactly the same as it was immediately *before* it, as far as the total number of *variations* is concerned. It is proved in the present proposition that immediately *after* each of these changes there must be a variation between X and $X_1$, which remains undisturbed till a root of $X = 0$ passes, when this variation is lost: thus the series of signs has one variation less than it had before this passage.

The next change in $X_1$ restores the variation between X and $X_1$, or rather introduces a new variation; but yet the lost variation is never recovered—the permanency that supplied its place can merely have shifted its situation; so that the successive changes of sign in $X_1$ do no more than prepare the way for a permanent loss of variation by the passage of a root of $X = 0$. No such loss could take place if a change in $X_1$ did not cause the signs of X and $X_1$ to be *unlike* before the passage of the root; nor when the loss *did* take place could it be permanent if a subsequent change in $X_1$, making the signs of X and $X_1$ again unlike, could thus add a variation to the total number. This is impossible by Proposition III.

(71.) We have hitherto considered real roots only to exist in the equation: let now imaginary roots be introduced; that is, suppose the polynomial X to be multiplied by another P in which $x$ so enters that whatever number we put for it, P always gives a *plus* result (21). Our original equation, which we shall suppose of the $n$th degree, will then be $PX = 0$; and to obtain from the first member of this the derived or limiting polynomial, let us remember that this polynomial is formed from the simple factors $(x - a)$, $(x - a_2)$, $(x - a_3)$, &c. composing the first member of the proposed equation, by multiplying every $n - 1$ of these together and adding the results (58). And this operation is obviously effected by multiplying *all* the factors of P by the limiting polynomial of X, then *all* the factors of X by the limiting polynomial of P, and adding the results: that is to say, calling the limiting polynomial of P, $P_1$, the limiting or derived polynomial of PX will be

$$PX_1 + P_1X$$

For brevity we shall call this derived polynomial X′.

Now as P never vanishes, by hypothesis, $P_1X$ necessarily vanishes when PX does, that is, when X does.

But as the entire expression X′ cannot vanish at the same time (Prop. II) it follows that the sign of X′ continues unchanged immediately before, at, and immediately after the passage of a root of $X = 0$.

*At* the passage, X′ is reduced to $PX_1$; and this, or simply $X_1$, continues also unchanged in sign immediately before, at, and immediately after the same passage, as already shown in the preceding demonstration.

Hence immediately before, and immediately after the passage of a real root of $PX = 0$, the signs of PX and its derived polynomial X′, are the very same as those X and $X_1$.

Consequently the conclusion already obtained remains undisturbed by the entrance of imaginary roots.

(72.) From what has now been established, we deduce the following precepts for determining the number and situations of the real roots of an equation:

1. With the given polynomial X, and the derived polynomial $X_1$, perform the process for finding the common measure, changing the signs of the remainders as they present themselves.

2. Substitute in the series of polynomials formed by X, $X_1$, and all these changed remainders, first $-\infty$ and then $+\infty$ for $x$; and count the number of variations *lost* in passing from $-\infty$ to $+\infty$. This will be the number of real roots in the equation.

3. Substitute also 0 for $x$, and count the number of variations lost in proceeding from 0 to $+\infty$. This will give the number of positive roots; while the variations lost in passing from $-\infty$ to 0 will make known the number of negative roots.

4. To determine the situations of the *positive* roots, substitute for $x$ in the series of polynomials the successive numbers 0, 1, 2, 3..... till the results furnish as many variations as were given by the substitution of $x = \infty$. The positive roots will all be comprised between 0 and the number last substituted; which last number will be the immediately superior integral limit of the positive roots, the several places of which will be discovered by counting the variations given by every consecutive pair of substitutions; for the *difference* in the number of variations will express exactly the number of roots lying between the two numbers whence these variations arise.

To determine the situations of the negative roots proceed as above with the successive substitutions 0, $-1$, $-2$, $-3$..... stopping as soon as the results furnish the same number of variations as were given by $x = -\infty$.

In employing the *ascending* series 0, 1, 2, 3.... for the detection of the positive roots, a variation will be *lost* every time a root lies between consecutive substitutions. In employing the *descending* series 0, $-1$, $-2$, $-3$.... a variation will be *gained* every time a root is passed. This is plain from the preceding demonstration.

(73.) To these precepts several useful particulars may be appended as immediate inferences from the foregoing investigations. We shall notice one or two.

1. If we omit any number of leading terms in the series $X, X_1, X_2 \ldots$ and, commencing with an advanced term $X_p$, consider only the remaining polynomials, then, since the evanescence of any one of these polynomials between $X_p$ and the last can produce no change in the number of variations so long as $X_p$ itself undergoes no change of sign (which is proved exactly as in art. 68), it follows that whenever, in deducing the complete series of polynomials, we arrive at one $X_p$ which we know will preserve the same sign whatever number be put in it for $x$, we may stop at this polynomial and disregard all that follow as useless for our purpose. For as, in the case supposed, the series

$$X_p,\ X_{p+1},\ X_{p+2}\ \ldots\ldots$$

furnishes the same number of variations, whatever value be put for $x$, the *difference* of the number of variations for any *two* substitutions $p$, $q$ will always be zero; and consequently the difference furnished by the complete series,

$$X, X_1, X_2, \ldots X_p, X_{p+1}, X_{p+2} \ldots\ldots$$

for any two substitutions, will be obtained by employing only the partial series

$$X, X_1, X_2 \ldots\ldots X_p$$

There is one case in which it will generally be easy to examine whether or not a polynomial $X_p$ preserves its sign unchanged throughout all values of $x$. It is the case in which $X_p$ is a *quadratic* expression; for if such an expression have the product of its extreme terms greater than one fourth the square of the mean, the sign of it will continue invariable for all values of $x$, but not else (*Algebra*, art. 107.) Under this condition the roots of the quadratic equated to zero are imaginary; and such must necessarily be the character of the roots in order that any polynomial may preserve its sign unchanged throughout all substitutions for $x$, this unchanged sign being invariably that of its first term (21.)

Even if we were to stop at a polynomial $X_p$ whose component simple factors are *real*, still the incomplete series might be turned to useful account in the analysis of the equation, provided we first determine the situations of the roots of $X_p = 0$. For if these situations be between the numbers

$$[a, b], \quad [e, f], \quad [k, l], \ldots\ldots [p, q]$$

$a$ being the greatest number and $q$ the least, then, since $X_p$ undergoes no change of sign in the interval $[\infty, a]$, the polynomials after $X_p$ are of no use in the search after roots in this interval. The same may be said of the intervals $[-\infty, q]$, $[b, e]$, $[f, k]$, &c.

2. The polynomial which immediately precedes the last in the series is always of the first degree in $x$, that is, it is of the form $ax + b$. This becomes zero for $x = -\frac{b}{a}$; and since the value which causes any polynomial to become zero causes those on

either side of it to take opposite signs (67), it follows that if we put $-\frac{b}{a}$ for $x$ in the *preceding* polynomial, and change the resulting sign, we shall obtain the proper sign of the *succeeding* expression, that is, of the constant number which closes the series; and as the *sign* is all we have occasion for, it is clear that we may employ this mode of deducing it, and dispense with the actual calculation of the final remainder itself.

(74.) In what has hitherto been said, the original equation $X = 0$ is supposed to have no equal roots. If we now consider equal roots to be introduced, then the polynomial X and all the following polynomials will have a common measure (62), as $X_p$, which will discover itself by the next polynomial $X_{p+1}$ becoming zero. This common measure, equated to zero, will involve some of the roots of $X = 0$; and $\frac{X}{X_p}$ equated to zero will involve the remaining roots. The number and situations of these remaining roots may, however, be discovered by employing X, $X_1$ and the other polynomials derived, in the usual way, as if $X = 0$ contained no equal roots; for the additional common factor $X_p$, which they all involve, will have no effect upon the variations furnished by the series of polynomials for any proposed value of $x$; for if this value of $x$ should render $X_p$ *plus*, the resulting *signs* will be the same as would have occurred had that factor not entered; and if the proposed value render $X_p$ *minus*, it will merely have the effect of *changing all the signs*, and this will not interfere with the number of variations.

Hence the rule applies whether equal roots enter or not; the only thing to be observed when equal roots enter is that the series of polynomials, derived in the ordinary way, will make known the number and situations of all the roots taken *singly*. And the common measure, with which this series terminates, when equated to zero, will furnish a new equation involving those remaining roots which, combined with the former, cause the *repetitions*.

(75.) There are some interesting truths connected with the entrance of equal roots into an equation which deserve to be mentioned. We believe, however, that they have never before been noticed.

1. If an equation have a pair of equal roots and no more, these roots must be commensurable; that is, they must be either integers or fractions. For our series of polynomials must in the case supposed have for a common measure, and terminate in, a polynomial of the first degree; and this equated to zero will furnish one of the equal roots. But this polynomial, $ax+b$, has necessarily finite coefficients. Consequently $x=-\frac{b}{a}$ is necessarily either a whole number or a finite fraction.

2. If the first coefficient in the proposed equation be unity, then the equal roots must be integral; because no fractional root can enter the equation (26).

3. If an equation have three equal roots and no more, these also must be commensurable. For in this case our series of polynomials must terminate in a quadratic, which equated to zero will give *two* of those roots. And we have just seen that these two must be commensurable: hence the three roots must be either integers or finite fractions. And if the first coefficient of the proposed be unity the roots must as before be integral.

4. Reasoning in this way we infer generally that if an equation have $m$ equal roots and no other groups of equal roots, these $m$ roots must be commensurable; and if the first coefficient of the equation be unity they must be integral.

5. When but a single pair of equal roots exists still both roots of the quadratic polynomial, that is of the polynomial preceding the last, must be commensurable. For one is commensurable by 1 above; and as the other is got by adding this to the coefficient of the second term, it must be commensurable also.

6. To these we may add the obvious truth that if a biquadratic equation have two pairs of equal roots the first member of it must be a complete square, viz. the square of the final quadratic polynomial at which the operation for the common measure stops.

# CHAPTER II.

## APPLICATION OF THE PRECEDING THEORY TO THE ANALYSIS OF CUBIC EQUATIONS.

(76.) FROM what has been established in the preceding chapter it appears that, in order to analyse an equation, we must perform the operation for the common measure upon the first member of that equation and of the limiting equation derived from it, changing the signs of the several remainders as they arise. The modified remainders thus obtained, in conjunction with the two expressions on which they depend, furnish a series of polynomials which make known, by the changes which their signs undergo for different values of $x$, how many roots of the equation those values comprise.

As the information sought from this series of polynomials thus depends entirely upon the *signs* and not upon the numerical *values* of them, for different substitutions for $x$, it is plain that we are at liberty to multiply or divide any of them we please by any *positive* quantity whatever; and therefore for the purpose of clearing away fractions from the polynomials, or with a view to preclude their entrance, in the process of derivation, we shall not scruple to avail ourselves of this privilege.

Let us commence with the general equation of the third degree, which we shall write

$$ax^3+bx^2+cx+d=0$$

and let us actually seek, by the ordinary operation, for a common measure between the first member of this, viz.

$$X=ax^3+bx^2+cx+d$$

and

$$X_1=3ax^2+2bx+c$$

the first member of its limiting equation (58). The first step of this operation is as follows, the dividend being multiplied by the positive number 3 to avoid fractions:

$$\begin{array}{rl} 3ax^2+2bx+c) & 3ax^3+3bx^2+3cx+3d\ (x \\ & \underline{3ax^3+2bx^2+\ cx\qquad\ } \\ & \qquad\qquad bx^2+2cx+3d \end{array}$$

This remainder is not that which terminates the step, as the division by the present divisor is not yet completed; nor will it be till we reach a remainder a unit lower in degree than that divisor. We have, therefore, another division to perform; and to effect this, without fractions, must multiply the remainder just obtained by $3a$.

The remaining part of the step will therefore be

$$\begin{array}{l} 3abx^2+6acx+9ad\ (b \\ \underline{3abx^2+2b^2x+\ bc\qquad} \\ (6ac-2b^2)x+9ad-bc\ \ldots\ [1] \end{array}$$

Thus the first quotient is $Q_1=x+b$, and the first remainder, with signs changed, is

$$X_2=(2b^2-6ac)\,x-9ad+bc\ \ldots\ [2]$$

The remainder [1] is, of course, obtained in the usual way; by adding the lower line, with signs changed, to the upper; the modified remainder [2] on the contrary is the upper line, with signs changed, added to the lower; that is it is the sum of

$$2b^2\,x+bc$$

and

$$-6ac\,x-9ad$$

or regarding the *coefficients* only, the sum of

$$b\times\ \overline{2b+c}$$

and

$$3a\times\overline{-2c-3d}$$

**Now these two lines may be obtained at once from the original polynomials $X$, $X_1$ without going through the whole process above. We shall merely have first to write down the coefficients of $X_1$ in a row, separated from one another by their proper signs; then, commencing with the second coefficient in $X$, to write this, minus twice the third, and minus 3 times the fourth, under the former row; and we shall thus have the two rows**

$$3a+2b+c$$

$$b-2c-3d$$

from which if the leading term in each be cut off for *multipliers*, and these multipliers combined, the lower with the upper row, and the upper with the lower row, the preceding results will obviously be obtained: thus

$$\begin{array}{r|l} 3a & +2b+c \\ b & -2c-3d \\ \cline{2-2} & \quad 2b^2+bc \\ & -6ac-9ad \\ \cline{2-2} & (2b^2-6ac)-9ad+bc \end{array}$$

$$\therefore\ X_2=(2b^2-6ac)x-9ad+bc$$

(77.) It is obvious that if the two multipliers $3a$ and $b$ had a common factor, that factor might be expunged; since, by preserving it, it would enter into all the terms of the result and would therefore be superfluous (p. 88). If $b$ be zero, that is if the proposed equation want the second term, then the multipliers are $3a$ and 0, or dividing by $3a$, they are 1 and 0. Hence the result consists simply of *once* $-2cx-3d$; so that in this case $X_2$ is obtained from $X$ as easily as $X_1$ is; for it is obtained simply by multiplying the last two terms of $X$ by $-2$ and $-3$ respectively.

Sometimes the result may be simplified, although $3a$ and $b$ have no common factor; such will be the case whenever the numbers $b$, $2c$, $3d$, in the lower row, or $c$, $2b$, $3a$, in the upper row, admit of a common factor; for this factor may obviously be neglected; since, as before, it would enter all the terms of the result.

And lastly, a like simplification may be effected if a common factor be found to enter all the four numbers to the right of the line which cuts off the multipliers, that is if $2b$, $c$, and $3d$ have a common factor.

(78.) For the convenience of reference it may be as well to express the foregoing operations in the form of a rule.

Rule 1. Arrange the coefficients of $X_1$ in a row with their proper signs.

2. Underneath these write in order the *second* coefficient of X, minus twice the third, minus three times the fourth.

3. Cut off the leading coefficient of each row, and multiply the lower of these by the remaining coefficients of the upper row, and the upper one by the remaining coefficients of the lower row, ranging the results under these same coefficients.

4. Add together the results thus obtained, and the coefficients of $X_2$, properly modified as to signs, will be exhibited.

NOTE.—When the second term is absent from the proposed equation, then $X_2$ will be obtained by multiplying the last two terms of that equation by $-2$ and $-3$ respectively.

It is also to be observed that the more to simplify the operation common factors should be expunged agreeably to the directions in (77).

(79.) As to the remainder $X_3$, which immediately follows $X_2$, and which is evidently the final one, it may be obtained as already remarked at p. 85, by substituting the value of $x$, deduced from $X_2=0$, in $X_1$; and changing the resulting sign. But, as the *sign* is all that we want, it may generally be easily discovered by roughly estimating the effect of this substitution in $X_1$, without the trouble of accurately computing the value.

We shall now proceed to particular examples.

### EXAMPLES.

1. Required the number and situations of the real roots of the equation

$$x^3-5x^2+8x-1=0$$

In this example

$$X=x^3-5x^2+8x-1$$

$$X_1=3x^2-10x+8$$

and the work for finding the next polynomial is, by the rule, as follows:

$$\begin{array}{r|r}
3 & -10+8 \\
-5 & -16+3 \\
\hline
 & 50-40 \\
 & -48+9 \\
\hline
 & 2-31 \\
\hline
\end{array}$$

$$\therefore X_2=2x-31$$

By equating this to zero we have $x=15\frac{1}{2}$; and it is plain that if this value of $x$ were put in $X_1$ the resulting sign would be plus. Hence the sign of $X_3$ is *minus*.

Arranging now the several polynomials in a row we have

$$\begin{array}{lccccl}
 & X & X_1 & X_2 & X_3 & \\
\text{For } x=-\infty \ldots & - & + & - & - & \text{two variations,} \\
x=+\infty \ldots & + & + & + & - & \text{one variation,}
\end{array}$$

consequently, as only one variation is lost between the limits

$-\infty$ and $+\infty$ we infer that the equation has only one real root. This real root being of necessity opposite in sign to the last term of the given equation (22) is *plus*. We may find its situation by means of X alone, without the aid of the other polynomials, by substituting the numbers 0, 1, 2, &c., successively in X, till we get consecutive results with opposite signs, remembering that these substitutions cannot possibly extend beyond the number 6 (44). For $x=0$ the result is $-$. For $x=1$ it is $+$. Hence the root lies between 0 and 1.

2. Required the number and situations of the real roots of the equation

$$4x^3+27x^2+18x-41=0$$

Here

$$X = 4x^3+27x^2+18x-41$$

$$X_1=12x^2+54x+18$$

or dividing by 6

$$X_1=2x^2+9x+3$$

and the operation for $X_2$ is as follows:

$$\begin{array}{r|rr} 2 & +9 & +3 \\ 27 & -36 & +123 \\ \cline{2-3} & 243 & +81 \\ & & \\ & -72 & +246 \\ \cline{2-3} & 171 & +327 \\ \cline{2-3} \end{array}$$

$$\therefore X_2=171x+327$$

We might have worked with smaller numbers by suppressing the factor 3, common to all the numbers in the second row, as recommended at (77). Making this reduction, the operation will be as follows:

$$\begin{array}{r|rr}
2 & +9 + & 3 \\
9 & -12 + & 41 \\
\hline
 & 81 + & 27 \\
 & -24 + & 82 \\
\hline
 & 57 + & 109 \\
\hline
\end{array}$$

$$\therefore X_2 = 57x + 109$$

This value of $X_2$ is the same as that to which the former reduces when divided by 3. Equating it to zero we have for $x$ the fraction $-\frac{109}{57}$ which it is easy to see must, when substituted in $X_1$, give a *minus* result. Hence $X_3$ is *plus*; and the polynomials are therefore

$$X = 4x^3 + 27x^2 + 18x - 41$$

$$X_1 = 2x^2 + 9x + 3$$

$$X_2 = 57x + 109$$

$$X_3 = +$$

Arranging these in a row we have

| | $X$ | $X_1$ | $X_2$ | $X_3$ | |
|---|---|---|---|---|---|
| For $x = -\infty$ . . . . | − | + | − | + | three variations |
| $x = +\infty$ . . . . | + | + | + | + | no variations |

Hence the three roots are all real.

| | | | | | |
|---|---|---|---|---|---|
| $x = 0$ . . . . . | − | + | + | + | one variation |

Consequently, as two variations are lost in passing from $-\infty$ to 0, there are two negative roots; and one being lost in passing from 0 to $+\infty$ there is one positive root.* Let us first find the places of the negative roots.

|  | $X$ | $X_1$ | $X_2$ | $X_3$ |  |
|---|---|---|---|---|---|
| $x=0$ | $-$ | $+$ | $+$ | $+$ | one variation |
| $-1$ | $-$ | $-$ | $+$ | $+$ | one variation |
| $-2$ | $-$ | $-$ | $-$ | $+$ | one variation |
| $-3$ | $+$ | $-$ | $-$ | $+$ | two variations |
| $-4$ | $+$ | $-$ | $-$ | $+$ | two variations |
| $-5$ | $+$ | $+$ | $-$ | $+$ | two variations |
| $-6$ | $-$ | $+$ | $-$ | $+$ | three variations. |

Hence one negative root lies between $-2$ and $-3$, and the other between $-5$ and $-6$. As to the positive root, that is found from the single polynomial X to lie between 0 and 1.

In this, as well as in the preceding example, we have determined the exact number of the real roots, before seeking their places, by actually putting $-\infty$ and $+\infty$ in the polynomials, for the purpose of discovering the variations lost between these extreme limits. But when, as is usual, these polynomials regularly descend in degree by unity, and have their leading signs all *plus*, as in the present example, it is plain that the substitution of $-\infty$ for $x$ will give nothing but variations, and the substitution of $+\infty$ nothing but permanencies; so that we may at once conclude, from an inspection of these leading signs, when the roots are all real, and when two of them are imaginary.

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* It should be remembered, however, that when we have found out that all the roots are real, the number of them that are positive or negative may be at once ascertained by the *rule of signs* from the original equation.

3. Required the analysis of the equation

$$x^3+11x^2-102x+181=0$$

Here

$$X_1=3x^2+22x-102$$

$$\begin{array}{r|rr} 3 & +\ 22 & -\ 102 \\ 11 & +\ 204 & -\ 543 \\ \hline & 242 & -1122 \\ & & \\ & 612 & -1629 \\ \hline & 854 & -2751 \\ \hline \end{array}$$

Hence the polynomials are

$$X\ =x^3+11x^2-102x+181$$

$$X_1=3x^2+22x-102$$

$$X_2=854x-2751$$

$$\therefore X_3=+$$

And as the leading signs are all positive, we infer that the roots are all real (p. 95). Hence, as there are two variations, and one permanence of sign in the original polynomial, we further infer from the rule of Descartes, that two of these roots are positive, and the other negative.

In order to ascertain the situations of the positive roots, we have

|  | $X$ | $X_1$ | $X_2$ | $X_3$ |  |
|---|---|---|---|---|---|
| For $x = 0$ . . . | + | — | — | + | two-variations |
| 1 . . . | + | — | — | + |  |
| 2 . . . | + | — | — | + |  |
| 3 . . . | + | — | — | + |  |
| 4 . . . | + | + | + | + | no variation. |

Hence the positive roots both lie between 3 and 4.

The negative root is found from X alone, to lie between —10 and —20.

It may be proper to remark, before proceeding further, that after having ascertained the total number of real roots, and before seeking their respective situations by the substitution of particular consecutive values for $x$, we always ascertain, by an application of one or other of the propositions in Chapter III. of the preceding section, the limit between which and 0, all the roots under examination must be comprised. In all the preceding examples we have thus ascertained that the roots whose places are sought necessarily lie between 0 and 10, or between 0 and — 10. And we have accordingly commenced our substitutions at 0, and have proceeded on with consecutive numbers towards the superior limit thus found with a full assurance that our analysis must become complete before this limit is passed.

But if the superior limit should turn out to be above 10, we should depart from this mode of proceeding; and instead of applying the scale of numbers 0, 1, 2, &c. or 0, —1, —2, &c. we should employ the scale 0, 10, 20, &c. or 0, —10, —20, &c. accordingly as we seek the places of the positive or of the negati roots. We should in this manner always learn within the c pass of which particular compartment, of ten grades in the a imetical scale, the roots are really situated; and then, comm cing with the first of these grades, and substituting it and those which follow successively, as in the foregoing examples, we should be sure of detecting the places of the roots within the range of *ten* substitutions. The following example will illustrate these remarks.

4. It is required to analyse the equation

$$3x^3 + 75x^2 - 117x - 280 = 0$$

Here

$$X_1 = 9x^2 + 150x - 117$$

or

$$X_1 = 3x^2 + 50x - 39$$

$$\begin{array}{r|l} \frac{1}{3} & +50 - 39 \\ \frac{25}{75} & +234 + 840 \\ \hline & 1250 - 975 \\ & 234 + 840 \\ \hline & 1484 - 135 \\ \hline \end{array}$$

Consequently

$$X = 3x^3 + 75x^2 - 117x - 280$$

$$X_1 = 3x^2 + 50x - 39$$

$$X_2 = 1484x - 135$$

$$\therefore X_3 = +$$

As the leading signs are all *plus*, we infer that the roots are all real. Hence, by the rule of Descartes, there are two negative roots and one positive root. The single positive root may detected from an examination of X only; we shall therefore ploy the preceding polynomials for the separation of the neg roots.

By changing the alternate signs of X we get

$$3x^3 - 75x^2 - 117x + 280 = 0$$

And by Prop. III, page 53, a superior limit to the positive roots of this equation is $\frac{117}{3} + 1$ or 40, and consequently an inferior limit to the negative roots of the proposed is —40. We shall therefore proceed with our analysis as follows:

| | $X$ | $X_1$ | $X_2$ | $X_3$ | |
|---|---|---|---|---|---|
| For $x = 0$ . . . | — | — | — | + | one variation |
| —10 . . . | + | — | — | + | two variations. |

Thus one of the two roots is comprised between 0 and —10, but only one

| | | | | | |
|---|---|---|---|---|---|
| —20 . . . | + | + | — | + | two variations |
| —30 . . . | — | + | — | + | three variations. |

Therefore the remaining root lies between —20 and —30. The place of this, therefore, is accurately determined; that is, its first figure is ascertained: it is —2, in the ten's place. But the situation of the other root, that between 0 and —10, is not yet discovered with sufficient precision, as we cannot at present pronounce upon its first figure. We must therefore be more minute in our analysis of the proposed equation within the limits of $x = 0$ and $x = -10$: that is, we must subdivide this interval as follows:

| | | | | | |
|---|---|---|---|---|---|
| $x = 0$ . . . | — | — | — | + | one variation |
| —1 . . . | — | — | — | + | |
| —2 . . . | + | — | — | + | two variations. |

the root sought lies between —1 and —2. Its first figure
refore —1 in the unit's place.

ught to be remarked, however, that when, as in the present
only a single root lies in the interval to be examined, the
tions need not be made in any polynomial but the first.

It is plain that we have to continue these substitutions only till X changes sign, when the situation of the root becomes known. And thus, in Example 2, the substitutions $-3$, $-4$, $-5$, $-6$ need have been made only in X; so that the other columns of signs, furnished by these substitutions, are superfluous.

5. It is required to analyse the equation

$$5x^3-8x^2+23x-17=0$$

Here

$$X_1=15x^2-16x+23$$

And it is easy to see, from an inspection of the coefficients of this expression, that four times the product of the extreme terms exceeds the square of the mean; so that (*Algebra*, p. 125) the roots of $X_1=0$ are imaginary, and therefore (73) the polynomials $X_2$, $X_3$ need not be computed.

For $x=-\infty$ the polynomials X, $X_1$ furnish a variation, and for $x=+\infty$ a permanency; there is therefore but one real root, which, from the sign of the final term in X, must be positive (22.)

Its situation found from the single polynomial X is between 0 and 1.

6. It is required to analyse the equation

$$5x^3-7x+11=0$$

Here

$$X_1=15x^2-7$$

and Note, p. 91

$$X_2=14x-33$$

$$\therefore X_3=-$$

Hence (p. 95) the equation has a pair of imaginary roots; and consequently, from the sign of the final term in X, the real root is negative (22). It is found from substituting 0, $-1$, $-2$, &c X to lie between $-1$ and $-2$.

7. It is required to analyse the equation

$$13x^3-45x-8=0$$

Here

$$X_1 = 39x^2 - 45$$

and NOTE, p. 91

$$X_2 = 90x + 24$$

$$\therefore\ X_3 = +$$

Hence the roots are all real (p. 95.) To determine their character we have

| | $X$ | $X_1$ | $X_2$ | $X_3$ | |
|---|---|---|---|---|---|
| For $x = 0$ . . . . | $-$ | $-$ | $+$ | $+$ | one variation |
| $\infty$ . . . | $+$ | $+$ | $+$ | $+$ | no variation. |

Hence the roots are, one positive and two negative.

If we complete the proposed equation by inserting the absent term $0x^2$, and then change alternate signs, the $-45$ will remain unaltered. Hence $\frac{45}{13} + 1$ is numerically greater than the greatest root whether positive or negative (44), so that our substitutions will not extend beyond the number 5.

To find the places of the negative roots we have

| | $X$ | $X_1$ | $X_2$ | $X_3$ | |
|---|---|---|---|---|---|
| For $x = 0$ . . . | $-$ | $-$ | $+$ | $+$ | one variation |
| $-1$ . . . | $+$ | $-$ | $-$ | $+$ | two variations |
| $-2$ . . . | $-$ | $+$ | $-$ | $+$ | three variations. |

Hence the negative roots are situated, one between 0 and $-1$: and the other between $-1$ and $-2$. Also the positive root is by substituting 0, 1, 2 . . . . in X, to lie between 1 and 2.

nay be observed here, as at the end of Example 4, that after ve found the substitution of $-1$ to produce a change of n X we have no absolute occasion for the other polynomials, as the substitution must continue till another change occurs in X, and no longer. In example 3, all the polynomials were indispensable throughout the whole series of substitutions.

8. Required the analysis of the equation

$$12x^3 - 120x^2 + 326x - 127 = 0$$

Here

$$X_1 = 36x^2 - 240x + 326$$

or

$$18x^2 - 120x + 163$$

Therefore, by the rule

$$\begin{array}{r|rrrr}
\overset{3}{18} & - & 120 & + & 163 \\
\overset{-20}{-120} & - & 652 & + & 381 \\ \hline
 & & 2400 & - & 3260 \\
 & - & 1956 & + & 1143 \\ \hline\hline
 & & 444 & - & 2117 \\ \cline{3-5}
\end{array}$$

Consequently

$$X = 12x^3 - 120x^2 + 326x - 127$$

$$X_1 = 18x^2 - 120x + 163$$

$$X_2 = 444x - 2117$$

$$\therefore X_3 = -$$

Hence two of the roots are imaginary (p. 95); and from the of the final term in X the real root must be positive.

It is easily seen, by a reference to this polynomial, to lie between 0 and 1.

Sometimes it will happen that the polynomial $X_2$ will terminate the series, not from the entrance of equal roots into the equation, but because of the coefficient of $x$ in $X_2$ becoming zero, thus reducing that expression to a constant number.

The following is an example of this.

9. Required the analysis of the equation.

$$x^3 + 15x^2 + 75x + 12 = 0$$

Here

$$X_1 = 3x^2 + 30x + 75$$

or

$$x^2 + 10x + 25$$

$$\begin{array}{r|rr}
1 & +\ 10 & +\ 25 \\
15 & -\ 150 & -\ 36 \\
\hline
 & 150 & +\ 375 \\
 & -\ 150 & -\ 36 \\
\hline
 & & 339 \\
\end{array}$$

Consequently the polynomials are

$$X = x^3 + 15x^2 + 75x + 12$$

$$X_1 = x^2 + 10x + 25$$

$$X_2 = 339$$

As $x = -\infty$ gives here only one variation, it follows that there can be but one real root. From the final sign of X this root must gative. It is obviously situated between 0 and $-1$.

Required the analysis of the equation

$$x^3 - 7035x^2 + 15262754x - 10000730880 = 0$$

As the coefficients in this example are very large, it will be advisable to transform the equation into another whose coefficients shall be smaller. This may be done by diminishing the roots by some convenient large number. A moment's consideration will suggest 1000 as such a number, or 2000 as one still better. We shall therefore transform the proposed equation into another, of which the roots may each be less than the roots of the original by 2000 (36.)

$$\begin{array}{rrrrl}
1 & -\,7035 & +\,15262754 & -\,10000730880 & (2000 \\
 & 2000 & -\,10070000 & 10385508000 & \\
\cline{2-2}\cline{3-3}
 & -\,5035 & 5192754 & & \\
 & 2000 & -\,6070000 & & \\
\cline{2-2}
 & -\,3035 & & & \\
 & 2000 & & & \\
\hline
\end{array}$$

$$x'^3 - 1035\,x'^2 - 877246\,x' + 384777120 = 0$$

$$\therefore\ X_1 = 3\,x'^2 - 2070\,x' - 877246$$

$$\begin{array}{r|rr}
\overset{1}{3} & -\,2070 & -\,877246\,^{*} \\
-\,\overset{-345}{1035} & +\,1754492 & -\,1154331360 \\
\cline{2-3}
\multicolumn{1}{r}{} & 714150 & +\,302649870 \\
\multicolumn{1}{r}{} & 1754492 & -\,1154331360 \\
\cline{2-3}
\multicolumn{1}{r}{} & 2468642 & -\,851681490 \\
\cline{2-3}
\end{array}$$

---

* The four numbers to the right of the multipliers in this example admit of division by 2. But the reduction arising from this division is trifling to justify the trouble of making it.

Hence the polynomials are

$$X = x'^3 - 1035x'^2 - 877246x' + 384777120$$

$$X_1 = 3x'^2 - 2070x' - 877246$$

$$X_2 = 2468642x' - 851681490$$

$$X_3 = +$$

Therefore the roots, or values of $x'$, are all real; and hence by the rule of signs two of them are positive and one negative. We thus learn that two of the roots of the original equations exceed 2000; and consequently to find the places of these we need only substitute in the polynomials just determined the numbers 0, 1000, 2000, . . . .

| | $X$ | $X_1$ | $X_2$ | $X_3$ | |
|---|---|---|---|---|---|
| For $x = 0$ . . . . | + | — | — | + | two variati |
| 1000 . . . . | — | | | | |
| 2000 . . . . | + | | | | |

Hence the positive roots of the equation in $x'$ lie one between 0 and 1000; and the other between 1000 and 2000. Consequently two roots of the proposed are between 1000 and 2000 and between 2000 and 3000, the first figure of the former of these being 1 in the place of thousands, and the first figure of the latter 2 in the place of thousands. Moreover, as all the roots of the proposed are real, we see by the rule of Descartes that the remaining root of that equation must also be positive.

As the sum of the two roots whose places have just been found must be below 5000, and as the coefficient of the second term with changed sign, that is 7035, is the sum of all the roots, it follows the remaining root must exceed 2000. Hence substituting 3000, . . . . . . for $x$ in the proposed equation we find this to lie between 3000 and 4000; so that the first figure of it is 3 place of thousands.

11. It is required to determine the conditions necessary in order that the equation

$$x^3+Ax+N=0$$

may have all its roots real.

$$X_1=3x^2+A$$

and Note (p. 91) $$X_2=-2Ax-3N$$

And substituting in $X_1$ the expression $-\frac{3N}{2A}$ for $x$ and changing sign, we have, after multiplying by $4A^2$ to remove the fraction,

$$X_3=-27N^2-4A^3$$

Now that all the roots may be real the leading signs of the polynomials, as well as the sign of $X_3$ must be *plus*; that is A must be negative, as also $27N^2+4A^3$. But for this alone to be negative it is evident that A must be so, because $27N^2$ is essentially positive. Hence the latter condition implies the former; so that the single condition

$$27N^2+4A^3<0$$

is sufficient to preclude the entrance of imaginary roots into the equation.

If

$$27N^2+4A^3=0$$

the equation must have a pair of equal roots (62), each of the equal roots being given by the equation $X_2=0$; that is each of these roots is

$$x=-\frac{3N}{2A}$$

These inferences were otherwise deduced at (60.)

(80.) The following examples are added as exercises fo student in the analysis of cubic equations :

12. $x^3-6x^2+9x-1=0.$

The roots are all real and situated as follows:

[0, 1]; [2, 3]; [3, 4]

that is between 0 and 1, between 2 and 3, and between 3 and 4.

13. $x^3-15x^2+63x-50=0$

Places of roots [1, 2]; [6, 7]; [7, 8].

14. $x^3-49x^2+658x-1379=0$

Places of roots [2, 3,]; [20, 30]; [20, 30].

15. $x^3-7x+7=0$

Places of roots [1, 2]; [1, 2]; [ — 3, — 4].

16. $x^3-6x^2+12x-1=0$

Place of root [0, 1]. Two roots imaginary.

17. $x^3-2x-5=0$

Place of root [2, 3]. Two roots imaginary.

18. $6x^3-141x+263=0$

Places of roots [2, 3]; [2, 3]; [— 5, — 6].

19. $4x^3-9x^2-36x+40=0$

Places of roots [0, 1]; [3, 4]; [— 2, — 3].

20. $x^3+30x^2+279x+811=0$

Places of roots [— 6, — 7]; [—8, — 9]; [— 10, — 20].

. $441x^3-6615x^2+32672x-52972=0$

Places of roots [3, 4]; [5, 6]; [5, 6].

(81.) In the general investigation (76) whence the preceding rule for the analysis of a cubic equation has been deduced, we have literally complied with the theory expounded in the former chapter; and have exhibited every part of the operation implied in the ordinary method of seeking the greatest common divisor of two algebraical expressions. But a slight inspection of the work will suffice to show that this method is encumbered with many superfluous appendages, which answer no other purpose than to complicate the several steps, and to give a distorted appearance to the entire arrangement.

So long as the method for finding the common measure of two polynomials was confined to the single purpose of reducing them to lower terms, and was thus an operation in but little request; the cumbrousness of the process excited scarcely any concern. But now that so important a problem as the complete analysis of numerical equations of all degrees is found to depend for its solution upon this same operation, it becomes matter of importance that the steps of it should be so modified and arranged as to furnish the several results or remainders with the utmost facility, whatever be the dimensions of the polynomials employed.*

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* It is remarkable however that the attention of analysts was not earlier awakened to the laborious character of the ordinary process for the common measure, and to the importance of an abridged method of conducting the operation; because in every theory for analysing an equation, this operation was always understood to be necessary as a mere preparative to the application of the proposed method. The only way of accounting for the great labour of this preparative process being overlooked, seems to be that the tediousness and difficulty of the analysis, even after this preparation was supposed to have been effected, were so formidable that, except for equations of a low degree, and with small coefficients, these methods were never submitted to actual practice; but were regarded as little other than matters of theoretical speculation. The process therefore of seeking the common measure was never brought into practical operation, except in cases too simple and easy to create any s
picion of the amount of numerical work which that process really invol
when the equation to be analysed reaches the fifth or sixth degree. T
labour was thus commonly spoken of by analysts as of very easy perfon

Now it is plain that these remainders may all be obtained without the actual insertion of the successive quotients; and indeed the formality of making each step an operation of *division*, thus requiring a divisor and a quotient, may be altogether dispensed with, and the process changed into one of *simple elimination*, like that employed in the solution of simultaneous simple equations (*Algebra*, art. 57). Thus the remainder

$$bx^2 + 2cx + 3d$$

obtained at (76) in the common way, is clearly no other than the result of eliminating the leading terms of the two expressions

$$ax^3 + bx^2 + cx + d$$

$$3ax^2 + 2bx + c$$

by the usual method of equalizing the coefficients of those terms,

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ance; and to clear an equation of equal roots, previously to the further analysis of it, was regarded in the same light as to clear it of fractions, or of its second term. The method of Sturm is wholly comprehended in this process for the common measure; and attention has thus been at length directed to the real difficulties connected with that operation. Whatever these difficulties may be, and they are only those inherent in long multiplications, it is plain that Sturm's theorem for analysing an equation must supersede every other method; unless indeed one be hereafter discovered that shall effect the analysis, whatever be the nature of the roots of the equation, without the aid of this same process for the common measure. No such method however exists at present; nor is it easy to conceive of the possibility of determining whether or not an equation has *equal roots*, without resorting to that process. For a discussion e comparative merits of the theories of Lagrange, Sturm, Budan, and rier, the student is referred to the *Mathematical Dissertations;* and e *Theory and Solution of Equations in general.*

or rather in this case, by equalizing the terms themselves; that is, by multiplying the first by 3, the second by $x$, and subtracting. The next remainder is got in the same way; that is, by equalizing the coefficients of the leading terms of

$$3ax^2 + 2bx + c$$

$$bx^2 + 2cx + 3d$$

and then subtracting. And it is obvious, whatever be the degree to which the highest power in each polynomial may rise, that this successive elimination of the leading terms involves all the *essential* operations of the ordinary method for finding the common measure of those polynomials. Instead, therefore, of disposing every pair of the polynomials as dividend and divisor, and seeking the quotient, we may arrange them one under the other, equalize the leading coefficients by the introduction of the requisite factors; and subtract the results of the multiplication. It is furthermore obvious that the *coefficients* alone are all that need enter the several steps; the proper powers of $x$ may be inserted afterwards, when the operation has terminated.

(82.) The leading coefficients in each pair of polynomials, and to which we see our attention is to be chiefly directed, may, we know, always be rendered equal by multiplying each by the other; so that the elimination may be effected by uniformly multiplying, by each leading coefficient, all the coefficients which follow the other, and then subtracting the results; which results, however, will have a superfluous common factor, if our multipliers, or leading terms, have one. Nevertheless, to give uniformity to the several steps of the work for finding the common measure, ac-cording to the principle of successive elimination here propo
we shall invariably cut off these leading terms as multipli

afterwards expunging whatever obvious common factor they may happen to involve, as in examples 4 and 7 given at pages 98, 102.

The operation at page 89 will thus be converted into the following

$$\begin{array}{r|l} \overset{1}{a} & +b+c+d \\ \overset{3}{3a} & +2b+c \\ \hline \end{array}$$

$$\begin{array}{l} 3b+3c+3d \\ 2b+c \\ \hline \end{array}$$

$$\begin{array}{r|l} b & +2c+3d \\ 3a & +2b+c \\ \hline \end{array}$$

$$\begin{array}{l} 6ac+9ad \\ 2b^2+bc \\ \hline 6ac-2b^2+9ad-bc \end{array}$$

And this when the proper powers of $x$ are inserted becomes

$$(6ac-2b^2)x+9ad-bc$$

as before. This expression, as well as all the others used in Sturm's theorem, is to have its signs changed; and it is desirable that, like as in the work of the preceding examples, the operation should itself furnish the results thus modified. This will be accomplished by merely transposing the rows of coefficients at the head of each step, changing the signs of those in the lower row, which are to right of the divisional line, and uniformly employing addition ead of subtraction. Thus the step just exhibited will be as ows:

$$\begin{array}{r|l}
\overset{3}{3a} & +2b+c \\
\overset{1}{a} & -b-c-d \\ \hline
\end{array}$$

$$\begin{array}{l}
2b+c \\
-3b-3c-3d \\ \hline
\end{array}$$

$$\begin{array}{r|l}
-b & -2c-3d \\
3a & -2b-c \\ \hline
\end{array}$$

$$\begin{array}{r}
-6ac-9ad \\
2b^2+bc \\ \hline
2b^2-6ac+bc-9ad
\end{array}$$

From the result deduced in this manner, our short rule would of course have followed as well as from the same result deduced by the more cumbersome process at (76). For the execution of the next following step of the operation, the step namely which furnishes the final remainder, we have no such short rule to offer; and on this account it was recommended in the analysis of a cubic equation to employ the property at (73) for the discovery of the *sign* of this final remainder, rather than to work up to it by prolonging the process for the common measure. Still, if the accurate calculation of this remainder be required, then we ought to work the step which leads to it by the method here proposed. For illustration, let us take the example at page 96, and subjoin to the work there exhibited, the supplementary operation which is required for the accurate determination of $X_3$; the whole will then be as follows:

$$X = x^3 + 11x^2 - 102x + 181$$

$$X_1 = 3x^2 + 22x - 102$$

$$\begin{array}{r|rr} 3 & +\ 22 & -\ 102 \\ 11 & +204 & -\ 543 \\ \hline & 242 & -1122 \\ & 612 & -1629 \end{array}$$

$$\begin{array}{r|rr} 854 & -2751 & \\ 3 & -\ \ 22 & +102 \\ \hline & -8253 & \\ & -18788 & +\ 87108 \end{array}$$

$$\begin{array}{r|r} -27041 & +\ 87108 \\ 854 & +\ \ 2751 \\ \hline & 74390232 \\ & -74389791 \\ \hline & 441 \end{array}$$

From the multiplications which are thus unavoidably involved in the last step, we see the expediency of evading the computation of that step by the application of the principle in (73), as in the examples already given. It should here be stated, however, that cases *may* occur in which the sign of the final remainder cannot be safely inferred from roughly estimating the effect of substituting $x$, as deduced from $X_2 = 0$ in the quadratic $X_1$. For should the quadratic give a result for this substitution, which differs from zero by only a very small number, it is plain that a rough estimate of this result would not enable us to determine with certainty whether it should be *plus* or *minus*, or whether indeed it might not be zero. Whenever, therefore, our rough estimate leaves us in uncertainty of this kind, the final result must be accurately uted, as above. The example we have here selected is more y in this predicament than any of the others given in this chapter. But we would recommend that for the sake of practice the student would now treat all the others as we have done this, and actually compute, in each case, the numerical value of $X_3$.*

(83.) It is not altogether irrelevant to remark here that the ordinary process of elimination in simultaneous simple equations may with advantage be modified by proceeding in a manner similar to that recommended above. Thus the example at p. 77 of the *Algebra*, if worked by the preceding principles, will stand as below. The equations proposed are

$$2x + 4y - 3z = 22$$

$$4x - 2y + 5z = 18$$

$$6x + 7y - z = 63$$

$$\begin{array}{r|rrl}
\overset{1}{2} & +4 & -\ 3 & = 22 \\
\overset{2}{4} & -2 & +\ 5 & = 18 \\
\hline
 & 8 & -\ 6 & = 44 \\
 & -2 & +\ 5 & = 18 \\
\cline{2-4}
 & 10 & -\ 11 & = 26 \quad \ldots\ldots [1] \\
\cline{2-4}
\end{array}$$

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* At the close of Chapter IV, which is in some measure supplemental to this and the one immediately following, will be found some useful inferences respecting the character of the roots of a cubic equation, as discoverable from inspecting its coefficients. But they are of little imp ance as regards the objects of the present chapter.

$$\begin{array}{r|rcrcl} \overset{1}{2} & +4 & - & 3 & = & 22 \\ \overset{3}{6} & +7 & - & 1 & = & 63 \\ \hline & 12 & - & 9 & = & 66 \\ & 7 & - & 1 & = & 63 \\ \hline \end{array}$$

$$\begin{array}{r|rcll} \overset{1}{5} & -8 & = & 3 & \ldots\ldots [2] \\ \overset{2}{10} & -11 & = & 26 & \\ \hline & -16 & = & 6 & \\ & -11 & = & 26 & \\ \hline & 5 & = & 20 & \ldots\ldots [3] \\ \hline \end{array}$$

$$\therefore 5z = 20 \quad \therefore z = 4 \text{ and } [2],\ 5y - 8z = 3 \quad \therefore y = 7$$

and from the first of the proposed equations $x = 3$.

The *space* occupied by the above work is certainly not *less* than that required by the common method (*Algebra*, p. 77); but the operations here exhibited are performed with great rapidity, and are marked by a uniformity and consecutiveness not observable in the ordinary process.

# CHAPTER III.

## ON THE ANALYSIS OF BIQUADRATIC EQUATIONS.

(84.) It has been shown in the preceding chapter that the ordinary operation for the common measure may be converted into the more brief and commodious process of successive elimination; and that the result of each complete step may be easily made to present itself with the changes of sign required by Sturm's Theorem.

In order to this we shall have to commence that step with the row of coefficients belonging to what in the old method would be the *divisor*, and to write those of the dividend underneath them; changing the signs of all the latter excepting the leading one, or that cut off for the multiplier. The results of the multiplications are then to be *added:* we thus obtain the first subordinate step. Underneath the sums so obtained we write the row of coefficients with which we commenced the step, changing as before all the signs after the first; which first, together with the number over it, is to be cut off, as above, for multipliers. The results of the multiplications being *added*, as before, furnish the next subordinate, or the *complete* step. All this will be abundantly intelligible from an inspection of the arrangement at page 112, and from reflecting that although in the instance there given the original polynomial is only of the third degree, yet it equally well illustrates the arrangement when the polynomial is of any higher degree, because the elimination of the *leading* terms cannot be affected by the number of subsequent terms.

That the several remainders come out with *changed signs* in arrangement referred to is easily seen to be a necessary con quence of that arrangement. For we here place the produ arising from the *dividend* below those arising from the *divis* which is the reverse of the arrangement in the common method; so that the subtraction of the lower row from the upper would necessarily give changed signs; and by securing a change of signs for the lower row, we have merely converted this subtraction into addition. Thus the numbers which close the first subordinate step have their signs all changed. The leading number of the row below these (that is, the lower multiplier) has, however, its own sign; so that the results of the multiplication necessarily have changed signs. But the upper multiplier, as well as the numbers which it multiplies, have all changed signs. The results of the multiplication here, therefore, have the proper signs; but by adding them instead of subtracting, these signs are virtually changed as well as those of the row above them. Hence the last result must come out with changed signs.

In the biquadratic equation

$$ax^4+bx^3+cx^2+dx+e=0$$

the arrangement of the first complete step must therefore be as follows:

$$\mathrm{X}=ax^4+bx^3+cx^2+dx+e$$

$$\mathrm{X}_1=4ax^3+3bx^2+2cx+d$$

$$\begin{array}{r|rrrr}
\overset{4}{4a} & +3b & +2c & +d & \\
\overset{1}{a} & -b & -c & -d & -e \\
\hline
 & 3b & +2c & +d & \\
 & -4b & -4c & -4d & -4e
\end{array}$$

$$\begin{array}{r|rrr}
-b & -2c & -3d & -4e \\
4a & -3b & -2c & -d \\
\hline
 & -8ac & -12ad & -16ae \\
 & 3b^2 & +2bc & +bd
\end{array}$$

The sum of these two rows completes the step. But these same rows are furnished by writing the row at the commencement, placing under these the second coefficient in $X_1$, minus twice the third, minus three times the fourth, and minus four times the fifth; and then cutting off the first term of each row for multipliers, and proceeding as in cubic equations, thus:

$$\begin{array}{r|rrr} 4a & +3b & +2c & +d \\ b & -2c & -3d & -4e \\ \hline & 3b^2 & +2bc & +bd \\ & -8ac & -12ad & -16ae \end{array}$$

If $b$ be zero, that is if the second term in the proposed equation be absent, then this last result is reduced to

$$-8ac-12ad-16ae$$

or, dividing by $4a$, to simply

$$-2c-3d-4e$$

Hence the rule for finding the coefficients of $X_2$ is as follows:

**Rule.** Write the coefficients of $X_1$ in a row; and as a second row write underneath them the second coefficient of $X$, minus twice the third, minus three times the fourth, and minus four times the fifth.

Cut off the leading coefficient in each row for multipliers; and proceed as in cubic equations for the determination of $X_2$.

To find $X_3$ continue the work as described in (82.)

**Note.** If the second term of the proposed equation be abse then the coefficients of $X_2$ are found immediately from those of by commencing at the third and multiplying them by $-2$, — and $-4$.

### EXAMPLES.

1. Required the analysis of the equation

$$x^4-2x^3-7x^2+10x+10=0$$

Here the polynomials are

$$\begin{aligned} X &= x^4-2x^3-7x^2+10x+10 \\ X_1 &= 2x^3-3x^2-7x+5 \end{aligned}$$

Hence by the rule

$$\begin{array}{rrrrrl}
\overset{1}{2} & \multicolumn{1}{|r}{-3} & -7 & +5 & & \\
\overset{-1}{-2} & \multicolumn{1}{|r}{+14} & -30 & -40 & & \\
\cline{2-4}
 & 3 & +7 & -5 & & \\
 & 14 & -30 & -40 & & \\
\cline{2-4}
 & 17 & \multicolumn{1}{|r}{-23} & -45 & & \\
 & 2 & \multicolumn{1}{|r}{+3} & +7 & -5 & \\
\cline{3-5}
 & & -46 & -90 & & \\
 & & 51 & +119 & -85 & \\
\cline{3-5}
 & & 5 & \multicolumn{1}{|r}{+29} & -85 & \\
 & & 17 & \multicolumn{1}{|r}{+23} & +45 & \\
\cline{4-5}
 & & & 493 & -1445 & \\
 & & & 115 & +225 & \\
\cline{4-5}
 & & & 608 & -1220 & \text{or } (\div 4),\ 152-305 \\
\cline{4-5}
\end{array}$$

Consequently the complete series of polynomials is

$$\begin{aligned} X &= x^4 - 2x^3 - 7x^2 + 10x + 10 \\ X_1 &= 2x^3 - 3x^2 - 7x + 5 \\ X_2 &= 17x^2 - 23x - 45 \\ X_3 &= 152x - 305 \\ \therefore \quad X_4 &= + \end{aligned}$$

As the leading signs of these are all *plus* we infer that the roots are all real; consequently by the rule of Descartes two of them are positive and two negative. A superior limit to the positive roots is 8; and, changing the alternate signs, an inferior limit to the negative roots is 5 (art. 47.) Hence our substitutions are confined within the limits 0, 8; and 0, —5.

For the positive roots we have

| | $X$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | |
|---|---|---|---|---|---|---|
| $x=0$ . . . . | + | + | — | — | + | two variations |
| 1 . . . . | + | — | — | — | + | |
| 2 . . . . | + | — | — | — | + | |
| 3 . . . . | + | + | + | + | + | no variation. |

Hence the two positive roots both lie between 2 and 3.

For the negative roots we have

| | $X$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | |
|---|---|---|---|---|---|---|
| $x=0$ . . . . | + | + | — | — | + | two variations |
| —1 . . . . | — | + | — | — | + | three variations |
| —2 . . . . | — | | | | | |
| —3 . . . . | + | | | | | |

Hence the two negative roots lie, one between 0 and —1; a
the other between —2 and —3.

2. Required the analysis of the equation

$$2x^4 - 13x^2 + 10x - 19 = 0$$

$$\text{Here } X = 2x^4 + 0x^3 - 13x^2 + 10x - 19$$

$$X_1 = 4x^3 + 0x^2 - 13x + 5$$

$\therefore$ Note, and $\div 2$, $X_2 = 13x^2 - 15x + 38$

And since it is obvious that $4\,(13 \times 38) > 15^2$, we shall have no occasion for the subsequent expressions (73.)

As $x = -\infty$ furnishes two variations, and $x = +\infty$, no variation, there are but two real roots: from the final sign in X, one of these must be positive and the other negative (23). By (47) the positive root cannot be so great as 8 nor (45) the negative root so small as $-11$. Substituting, therefore, 0, 1, 2 . . . in X, we find the former to lie between 2 and 3; and substituting 0, $-1$, $-2$ . . . we find the place of the latter to be between $-3$ and $-4$.

3. Required the analysis of the equation

$$x^4 - 3x^3 + 12x^2 - 13x + 21 = 0$$

$$X_1 = 4x^3 - 9x^2 + 24x - 13$$

$$\begin{array}{r|rrr}
4 & -9 & +24 & -13 \\
-3 & -24 & +39 & -84 \\
\hline
 & 27 & -72 & +39 \\
 & -96 & +156 & -336 \\
\hline
 & -69 & +84 & -297
\end{array}$$

Here again, since $4\,(69 \times 297) > 84^2$, we require only the three polynomials

$$X = x^4 - 3x^3 + 12x^2 - 13x + 21$$

$$X_1 = 4x^3 - 9x^2 + 24x - 13$$

$$X_2 = -69x^2 + 84x - 297$$

from which we infer, by a mere inspection of their leading signs, that all the roots of the proposed equation are imaginary.

4. Required the analysis of the equation

$$x^4 - 4x^3 \quad - 3x \ + 23 = 0$$

$$X = x^4 - 4x^3 \ + 0x^2 - 3x + 23$$

$$X_1 = 4x^3 - 12x^2 + 0x \ - 3$$

$$\begin{array}{r|llll}
\overset{1}{4} & -12 & +0 & -3 \\
\overset{-1}{-4} & -\ 0 & +9 & -92 \\
\hline
 & 12 & -0 & +3 \\
 & 0 & +9 & -92 \\
\end{array}$$

$$\begin{array}{r|llll}
{}_{3}\,12 & +9 & -89 & & \\
{}_{1}\,4 & +12 & -\ 0 & + & 3 \\
\hline
 & 9 & -89 & & \\
\end{array}$$

$$\begin{array}{r|lll}
 & 36 \ -\ 0 & + & 9 \\
\hline
{}_{15}\,45 & -89 & + & 9 \\
{}_{4}\,12 & -\ 9 & + & 89 \\
\hline
 & -356 & + & 36 \\
 & -135 & + & 1335 \\
\hline
 & -491 & + & 1371 \\
\end{array}$$

Hence the polynomials are

$$X = x^4 - 4x^3 \ + 0x^2 - 3x + 23$$

$$X_1 = 4x^3 - 12x^2 + 0x \ - 3$$

$$X_2 = 12x^2 + \ 9x - 89$$

$$X_3 = -491x + 1371$$

$$\therefore X_4 = -$$

Consequently there are but two real roots, and the substitutions of $x=0$ and $x=+\infty$ show that these are positive; and by (44) they must each be less than 5.

|  | X | $X_1$ | $X_2$ | $X_3$ | $X_4$ |  |
|---|---|---|---|---|---|---|
| $x=0$ .... | + | − | − | + | − | three variations |
| 1 .... | + | − | − | + | − |  |
| 2 .... | + | − | − | + | − |  |
| 3 .... | − |  |  |  |  |  |
| 4 .... | + |  |  |  |  |  |

Agreeably to what has been already stated (p. 101), whenever we are seeking the places of two roots, previously ascertained to be real, the polynomials beyond the first become useless as soon as a change in the sign of that first takes place. For this change indicates the place of one of the two roots, viz. the least; and by substituting on, towards the superior limit, we must arrive at another change marking the place of the remaining root; so that the changes in the subsequent polynomials furnish no additional information. The places of the roots in the preceding example are [2, 3] and [3, 4].

5. Required the analysis of the equation

$$x^4-27x^3+162x^2+356x-1200=0$$

$$X_1=4x^3-81x^2+224x+356$$

$$
\begin{array}{r|rrr}
4 & -81 & +224 & +356 \\
-9 & -108 & -356 & +1600 \\
-27 & -324 & -1068 & +4800 \\
\cline{2-4}
 & 729 & -2016 & -3204 \\
 & -432 & -1424 & +6400 \\
\hline
\end{array}
$$

$$
\begin{array}{r|rrr}
297 & -3440 & +3196 & \\
4 & +81 & -224 & -356 \\
\cline{2-4}
 & -13760 & +12784 & \\
 & 24057 & -66528 & -105732 \\
\hline
\end{array}
$$

$$
\begin{array}{r|rr}
10297 & -53744 & -105732^{*} \\
297 & +3400 & -3196 \\
\cline{2-3}
 & -399762 & -7860601 \\
 & 8752450 & -8227303 \\
\hline
 & 8352688 & -16087904
\end{array}
$$

Hence the polynomials are

$$X = x^4 - 27x^3 + 162x^2 + 356x - 1200$$

$$X_1 = 4x^3 - 81x^2 + 224x + 356$$

$$X_2 = 297x^2 - 3440x + 3196$$

$$X_3 = 8352688x - 16087904$$

$$\therefore X_4 = +$$

---

[*] The four numbers to the right of the divisional line in this step are all divisible by 4, the results of this division are

$$
\begin{array}{rr}
-13436 & -26433 \\
850 & -799
\end{array}
$$

and we have therefore employed these to complete the next step (77.)

Thus the roots are all real; and, consequently, by the rule of Descartes, three of them are positive, and one negative. The superior limit to the positive roots, found by (44), is 28. We shall therefore first seek these roots in the intervals 0, 10; 10, 20; 20, 30.

| | $X$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | |
|---|---|---|---|---|---|---|
| $x = 0$ . . . | $-$ | $+$ | $+$ | $-$ | $+$ | three variations |
| 10 . . . | $+$ | $-$ | $-$ | $+$ | $+$ | two variations |
| 20 . . . | $+$ | $+$ | $+$ | $+$ | $+$ | no variation. |

Hence one of the roots lies between 0 and 10, and the other two between 10 and 20; so that the first figure of each of these latter is 1 in the tens' place. We have now to determine more accurately the place of the other, by substituting in the polynomial X the numbers 0, 1, 2, . . . . This place we thus find to be between 2 and 3.

By changing the alternate signs of X, we find by (47) $-8$ for an inferior limit to the negative root. Hence, putting 0, $-1$, $-2$, $-3$.... for $x$ in X, we discover $-3$ to be the exact root, since it reduces X to zero.

## EXAMPLES FOR EXERCISE.

6. Required the analysis of the equation

$$x^4 + x^3 - 15x^2 - 19x - 3 = 0$$

The places of the roots are [3, 4]; [0, $-1$]; [$-1$, $-2$]; [$-3$, $-4$].

7. Required the analysis of the equation

$$x^4 - 12x^2 + 12x - 3 = 0$$

Places of roots [0, 1]; [0, 1]; [2, 3]; [$-3$, $-4$].

8. Required the analysis of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$$

Places of roots [0, 1]; [2, 3]; [5, 6]; [0, — 1].

9. Required the analysis of the equation

$$x^4 - 3x^2 - 30x - 88 = 0$$

One root $= 4$; one between $-2$ and $-3$; and two imaginary.

10. Required the analysis of the equation

$$x^4 + 6x^3 - 18x + 9 = 0$$

Places of roots [0, 1]; [1, 2]; [— 2, — 3]; [— 5, — 6].

11. Required the analysis of the equation

$$2x^4 + 5x^3 - 113x^2 - 176x - 240 = 0$$

Places of roots [7, 8]; [— 8, — 9]; two imaginary.

12. Required the analysis of the equation

$$2x^4 - 1400x^3 - 5x^2 - 11615x + 299145 = 0$$

Places of roots [1, 2]; [700, 800]; [— 1, — 2]; [— 70, — 80].

# CHAPTER IV.

## [SUPPLEMENTARY TO THE FOREGOING CHAPTER.]

(85.) IN all the preceding examples, as in those of Chapter II, the sign of the final remainder is ascertained by estimating the effect of substituting in $X_2$ the value of $x$ deduced from $X_3 = 0$. As already remarked (p. 113), when this value of $x$ is very nearly equal to a root of $X_2 = 0$ the result spoken of will be so near to zero that it may be impossible to determine with certainty the sign of it without accurate computation. In such a case, therefore, we may, as directed at p. 112, actually work up to the final result, when the proper sign will of course be exhibited.

Instances of the kind here noticed must always occur when the proposed equation has roots very nearly equal to one another; because then the simple expression $X_3$ is very nearly a common measure of all the preceding polynomials, and thus very nearly a divisor of $X_2$. But the roots of $X_2 = 0$ and $X_3 = 0$ may both occupy an interval where no roots of $X = 0$ exist; and as the character of this interval depends on the final sign, we must not trust to conjecture for the discovery of it. As in such circumstances the original coefficients may be large numbers, and those which have to be employed in the final step still larger, the working of this step will frequently involve long multiplications. Now there is another mode of arriving at the information sought, which will dispense with these multiplications, and which will frequently be much the shorter route to our object. This mode is to develope simultaneously the root of $X_3 = 0$, and the root, commencing with the same figure, of $X_2 = 0$; stopping the operations as soon as the two roots begin to differ. We shall then immediately see which of the two expressions $X_2$, $X_3$ reduces to zero first, or what sign $X_2$ takes when $X_3$ vanishes; and thence infer the sign of $X_4$.*

The student is not presumed, at this stage of his progress, to be acquainted with the easy and rapid process by which the root spoken of, of $X_2 = 0$ may be developed figure by figure. This will be taught in next chapter. But he may be expected to understand how such a development, if carried on simultaneously with the development of $x$ in $X_3 = 0$, will lead, without the expenditure of a single unnecessary figure, to the discovery of our final sign. The development of the root of $X_3 = 0$ is of course merely an operation of common division; when a figure or two of this is obtained then we are to leave it, and determine the root of $X_2 = 0$ to the same extent; after which we are to deduce figure after figure, first of one root and then of the other, alternately, till the figures cease to be the same for both. We shall give a practical illustration of this method of procedure hereafter. †

(86.) In a manner similar to that by which we may thus avoid the actual calculation of $X_4$, may we evade the computation of $X_3$ even. For, having reached the quadratic polynomial $X_2$, we can easily discover the limits between which each of its roots lies; then if either of the roots of $X_1$ are seen to lie within the same limits, or if it be matter of doubt whether a root be so situated, we may proceed to develope *each* root, in the common interval, till the figures differ; and thus determine what sign $X_1$ takes when $X_2$ vanishes, and thence infer the sign of $X_3$. For values of $x$ *beyond* the limits of the roots of $X_2 = 0$, the sign of $X_3$ is not wanted; the polynomials $X$, $X_1$, $X_2$ are the only ones necessary for our search after roots in those regions (73.)

(87.) It may happen that the roots of the quadratic $X_2 = 0$ are themselves very nearly equal, or comprised between limits so close as to separate only at a remote figure, between which limits two roots of $X_1 = 0$ may also lie. Such will be the case when the equation has three roots very nearly equal to one another. If we wish to avoid the long multiplications which enter into the determination of $X_3$ and $X_4$ in such a case as this, by adopting the course suggested above, we must develope both the neighbouring roots of $X_1 = 0$, as well as those of $X_2 = 0$, sufficiently to discover to us the order in which these expressions vanish in substituting continuously from one of the predetermined limits of the roots of $X_2 = 0$ up to the other. If, however, we wish to avoid the computation of $X_4$ only by the regular process, then we shall have to develope the two roots of $X_2 = 0$ (one being always got from the other by subtracting this other from the coefficient of the second term with changed sign) and the single root of $X_3 = 0$, in order to ascertain which expression vanishes first in passing continuously from one limit to the other.

---

* The two roots thus developed will be found to approximate much closer to each other than the nearly equal roots of the original equation do. This arises from the former roots being each situated between the two latter, and consequently closer together. See the last chapter of Sect. III.

† See the chapter on the Solution of Quadratic Equations in Section III.

(88.) What we have here stated is of perfectly general application; and is not exclusively confined to the classes of equations more immediately considered in the present work. From the character of the investigations in this and the preceding chapter, whence the rules for the analysis of numerical equations of the third and fourth degrees have been derived, it is obvious that these same rules admit of extension to all classes of numerical equations whatever; so that what has been stated above about the practicability of analysing the equation so soon as the quadratic remainder, or the subsequent remainder of the first degree is arrived at, is equally unrestricted in its application.

In fact we may stop at any remainder we please, provided only we possess the means of discovering the order in which the roots of this polynomial are interposed among those of the preceding polynomial. Sometimes these means offer themselves very readily; lthough, as remarked above, when roots very nearly equal enter ıe proposed equation, the order in which the roots of the two olynomials adverted to succeed each other will not be discoverble without we actually solve both equations. And this we must of course do whenever the indications, which always accompany such nearly equal roots, present themselves; although the result of our examination may show that the roots bearing these indications are nevertheless imaginary.

(89.) The better to illustrate what has just been said, as far at least as equations of the fourth degree are concerned, and to show the advantage that may be gained by stopping at the quadratic remainder, we shall revert to the examples already solved by the direct method.

1. In example 4 the polynomials, as far as the quadratic, are

$$\begin{aligned} X &= x^4 - 4x^3 + 0x^2 - 3x + 23 \\ X_1 &= 4x^3 - 12x^2 + 0x - 3 \\ X_2 &= 12x^2 + 9x - 89 \end{aligned}$$

Now from the rule of Descartes, or rather of Harriot, we infer that the equation $X = 0$ must have two imaginary roots, the nugatory term $0x^2$ being posited between *like* signs (29). Moreover by the same rule no negative roots can enter the equation, there being no permanencies of sign. And from (44) 5 is a superior limit to the positive roots.

The positive root of $X_2$ obviously lies between 2 and 3; so that $X_2$ preserves an invariable sign for all values of $x$ from $x = 2$ down to $x = 0$; and from $x = 3$ up to $x = +\infty$ (19); but of the latter interval we need only examine the portion between $x = 3$ and $x = 5$.

| | $X$ | $X_1$ | $X_2$ | |
|---|---|---|---|---|
| $x = 0$ . . . . | + | − | − | no root. |
| 1 . . . . | + | − | − | |
| 2 . . . . | + | − | − | |
| 3 . . . . | − | | | |

Having obtained this change of sign we may disregard all the polynomials except the first in our subsequent substitutions as at p. 123. We could not foresee, however, at the outset that we should arrive at such a change of sign; for the two roots might, for aught we could tell, lie between two consecutive numbers as between 0 and 1; or between 1 and 2. But from the preceding principles we are enabled to affirm that no root can exist between 0 and 2 and can therefore search with confidence beyond that limit.

2. As a second example let us take the 5th at page 123, which involves rather large numbers towards the close of the operation.

The polynomials up to the quadratic are as follow

$$X = x^4 - 27x^3 + 162x^2 + 356x - 1200$$

$$X_1 = 4x^3 - 81x^2 + 224x + 356$$

$$X_2 = 297x^2 - 3440x + 3196$$

The roots of the quadratic $X_2 = 0$ are readily discovered to lie between 1 and 2, and between 10 and 20. But for this latter root much narrower limits are very easily obtained, because, as we know that the sum of both roots is $\frac{3440}{297}$, which is plainly below 12, and that the former root exceeds 1, we infer that the latter root must lie between 10 and 11. Hence $X_2$ preserves its sign unchanged for all values of $x$ between the limits $x = -\infty$ and $x = 1$; between the limits $x = 2$ and $x = 10$, and between the limits $x = 11$ and $x = +\infty$. Let us then examine the proposed equation for these intervals

| | $X$ | $X_1$ | $X_2$ |
|---|---|---|---|
| $x = -\infty$ . . . . | + | − | + |
| 0 . . . . | − | + | + |
| 1 . . . . | − | + | + |

Hence there is one negative root and one only; and no positive root between 0 and 1.

| | | | |
|---|---|---|---|
| $x = 2$ . . . | − | + | − |

Here it is plain, from a glance at $X_1$, that this polynomial has continued plus throughout the interval [1, 2] whilst $X_2$ has changed; hence the state of the signs, in the series $X_1, X_2, X_3$, immediately before and after this change must be (67)

$$\begin{array}{ccc} & & X_3 \\ + & + & \\ & & - \\ + & - & \end{array}$$

so that no variation is lost in the passage over this root of $X_2 = 0$, in the series $X_1, X_2, X_3$; and we have seen above that no variation is lost in the former part of the series, $X, X_1$. Therefore there is no root between 1 and 2.

$$x = 10 \ldots\ldots + \quad - \quad -$$

Thus there is one positive root between 2 and 10, and one only.

$$11 \ldots\ldots + \quad - \quad +$$

Here again, as $X_1$ has continued *minus* throughout the interval [10, 11] it follows (67) that when $X_2$ vanished, $X_3$ took the sign *plus;* so that, including this sign, there were two variations immediately before, and two immediately after, the evanescence of $X_2$. Hence no variation is lost by the passage over the second root of $X_2 = 0$.

Again, for $x = \infty$ we have

$$x = \infty \ldots\ldots + \quad + \quad +$$

Thus two variations are lost in the interval [11, $\infty$]. Hence the equation has two more positive roots above 11. To find the places of these put $x = 20, 30, \ldots$

$$x = 20 \ldots\ldots + \quad + \quad +$$

Therefore, as two variations are lost in the interval [11, 20], we conclude finally that these two roots each commence with 1 in the tens' place. The single negative root is found as at p. 121; and thus is the equation completely analysed.

We may remark here, that if we had had any doubt as to whether the sign of $X_1$ continued constant throughout the interval [10, 11] we might have suspended our decision as to the roots in this interval, and have proceeded on to the interval [11, 20] which we see comprises two roots; and as two roots, one positive and one negative, were detected before, we should thus have analysed the equation without regarding the above interval.

But there can be no doubt as to the permanency of the sign of $X_1$ in the interval referred to, because if any changes had taken place *two* must have occurred; but two had already taken place, so that two more are impossible. In a similar manner the permanency of sign of $X_1$, between the limits [1, 2], is seen to be absolutely necessary because of the changes between $[-\infty, 0]$ and [0, 10].

More space is occupied in completing the analysis of the equation as above than the direct method seems to require; but this arises from the detail into which we have thought it instructive to the student to enter at each step. In reality the former method is by far the more lengthy and laborious.

The numbers involved in the first example at page 119 are so small that the several results are obtainable by the direct method as expeditiously as can be desired. We shall, however, for the purpose of additional illustration, analyse the equation by this second mode of proceeding.

3. The first three polynomials are

$$X = x^4 - 2x^3 - 7x^2 + 10x + 10$$

$$X_1 = 2x^3 - 3x^2 - 7x + 5$$

$$X_2 = 17x^2 - 23x - 45$$

The roots of $X_2 = 0$ are easily seen to lie between 2 and 3, and between $-1$ and $-2$. Hence $X_2$ preserves its sign unchanged from $x = -\infty$ to $x = -2$; from 0 to 2; and from 3 to $+\infty$

| | X | $X_1$ | $X_2$ | |
|---|---|---|---|---|
| $x=-\infty$ .... | + | − | + | no root |
| $-2$ .... | + | − | + | |
| $-1$ .... | − | + | −[*] | |
| $x=0$ .... | + | + | − | |

We need not attend here to the polynomials beyond X, as by the rule of Descartes there cannot be more than two negative roots, and two are now detected

| | | | |
|---|---|---|---|
| $x=2$ .... | + | − | − |

There is no root between 0 and 2

| | | | |
|---|---|---|---|
| $x=3$ .... | + | + | + |
| $=\infty$ .... | + | + | + |

Hence the only remaining interval in which roots can lie is that between 2 and 3. But to determine whether two roots, or none at all, are situated here, requires that we know the sign of $X_3$ for that value of $x$ which causes $X_2$ to vanish; and this can be discovered only from the sign which $X_1$ takes for the same value, or, which amounts to the same thing, only from knowing which of the two $X_1$ or $X_2$ vanishes first in proceeding continuously from 2 to 3. We have already explained that we may arrive at this knowledge by developing those two roots of $X_2=0$ and $X_1=0$ which commence with the common figure 2. But as we have not yet taught how to effect this development, we shall diminish each root by this common figure, which indeed is virtually the same thing; for if we in this way exhaust all the figures common to each root, the remaining parts of the two will commence with different figures, which when determined will make known which of the diminished roots lies nearest to 2, and consequently which of the original roots.

---

[*] We may conclude at once that but a single root lies in this interval, because there must be either *one* or *three*; and three negative roots are forbidden by the rule of Descartes.

Diminishing then by 2 we have (37)

$$\begin{array}{rrrrl} 2 & -3 & -7 & +5 & (2 \\ & 4 & 2 & -10 & \\ \hline & 1 & -5 & & \\ & 4 & 10 & & \\ \hline & 5 & & & \\ & 4 & & & \end{array} \qquad \begin{array}{rrrl} 17 & -23 & -45 & (2 \\ & 34 & 22 & \\ \hline & 11 & & \\ & 34 & & \end{array}$$

$$X'_2 = \underline{17x'^2 + 45x' - 23}$$

$$X'_1 = \underline{2x'^3 + 9x'^2 + 5x' - 5}$$

| | $X'_1$ | $X'_2$ |
|---|---|---|
| $x' = 0$ . . . | − | − |
| ·1 . . . | − | − |
| ·2 . . . | − | − |
| ·3 . . . | − | − |
| ·4 . . . | − | − |
| ·5 . . . | − | + |

We thus see that $X_2$ vanishes before $X'_1$ and consequently $X'_2$ vanishes before $X_1$. Hence $X_1$ continues *minus* while $X_2$ passes from *minus*, through *zero* to *plus;* so that (67) the signs of $X_1$, $X_2$, $X_3$, immediately before and after the passage, are

| $X_1$ | $X_2$ | $X_3$ |
|---|---|---|
| − | − | |
| | | + |
| − | + | |

Consequently a variation between $X_2$ and $X_3$ is lost in this passage; and this is the only loss the series $X_2$, $X_3$, $X_4$ can suffer in passing from $x = 2$ to $x = 3$, because $X_2$ passes but *once* through zero in that interval (73.) But in passing over the same interval a variation is lost in the former part of the series, that is in

$X, X_1, X_2$. Therefore on the whole *two* variations are lost: hence there are two real roots between 2 and 3.

The foregoing substitutions for $x'$, though involving but very little trouble, would have been much diminished in number if we had determined the second figure of each root by actual development. It may be further remarked that the analysis of the equation, by the above method, is at least as short as that by the direct rule; although, from the smallness of the coefficients, the example is a peculiarly favorable one for that rule; for the preceding analysis of the interval [2, 3] is all that should be compared with the work of determining $X_3$ and $X_4$ at p. 119, as the remainder of the analysis is much the same by either method.

It cannot have escaped notice, as an interesting feature of the method above explained, that the places of those roots of the proposed equation which lie *without* the limits which inclose the roots of the quadratic, may always be detected with great ease; and when our equation is to be solved, these roots may be developed, and the depressed equation, containing the other roots, thence deduced. When this depressed equation is a quadratic, all the roots become determinable without any further analysis of the equation proposed.

We would now strongly recommend to the student that he return to the examples proposed at page 125, and analyse them by the method here proposed, employing only the first step of the direct rule for the determination of $X_2$. In the chapter on the *Solution* of Biquadratic Equations we shall apply this method to an example of unusual difficulty—one which has two roots so nearly equal as to separate from each other only at the sixth decimal place. But for a more comprehensive discussion of difficulties of this kind, in connexion with equations of the higher degrees, the enquiring student is referred to the second edition of the *Theory and Solution of Equations in general,* and to the disquisition on Sturm's Theorem in the *Mathematical Dissertations.*

(90.) We shall merely remark here in conclusion, that from the foregoing analyses, it is obvious that the computation of the last two of Sturm's remainders may always be avoided, and the situations of the roots ascertained by a method of procedure involving, in most cases, much less numerical labour. We do not affirm that this method will *always* prove the shortest; because in those instances, unusual ones no doubt, where roots, whether real or imaginary, are so related as to cause those of the cubic and quadratic remainder to separate from one another only at a very remote figure, the twelfth or fourteenth for example; we say, in these unusual cases, the development of the two latter roots, up to the point of separation, may incur as much numerical work as the two steps of the direct rule which that work is intended to supersede.* It is worthy of notice, however, and is a circumstance greatly in favour of the direct method, that it is a process very simple and uniform in its character, very easily remembered, and universally effective in its results. It requires no collateral aid in the course of its operation, either for the purpose of testing, or of verifying its conclusions; it is independent even of the rule of Descartes; nor is it easy to conceive of a method for analysing an equation, applicable at once to all cases, that shall make so small a demand upon the algebraic knowledge and address of the computer as the Theorem of Sturm. The process which, in the present chapter, we have substituted for the two closing steps of that theorem, however much common multiplication it may occasionally save, is certainly not independent of these aids; as the operation is facilitated by whatever collateral information may offer itself as to the constitution of the proposed equation, and that of the derived cubic. Any knowledge obtained from external sources about the character of the roots of this latter equation more especially, is useful in this way. And on this account we shall terminate these remarks with a brief notice of certain criteria furnished by the coefficients of a complete equation of the third degree for testing, to a certain extent, the character of the roots.

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* It is proper to mention too, that when the roots are accurately *equal*, the developments spoken of *may* never reach a point of separation. This, however, cannot happen if the coefficient of the leading term in the proposed equation be unity; for then these developments must be finite and *integral* (75.) When the leading coefficient is not unity, such development must be a finite fraction, although the decimal form of it may be interminable. All equations may be disencumbered of the leading coefficient by (34); and it is especially deserving of remark, that whenever in such an equation two roots are discovered to lie in a known interval, and we are uncertain from the indications of their extreme proximity whether they are equal or not, our doubt may be readily removed by developing the roots lying between them of the derived cubic and quadratic; if these turn out to be equal integers, the roots in question are so, and they be- me thus determined; but if we arrive at a decimal in either develop- ent, the roots are necessarily unequal. Thus the development of the adratic alone will often speedily remove the uncertainty.

From

$$X = x^3 + bx^2 + cx + d$$

we have

$$X_1 = 3x^2 + 2bx + c$$

and by the rule (78)

$$X_2 = 2(b^2 - 3c)x + bc - 9d$$

And from these expressions we may deduce the following inferences, viz.

(91.) 1. If $3c > b^2$ two roots will be imaginary, for either of the two following reasons, viz., 1, because the roots of $X_1 = 0$ will be imaginary; or 2, because then the leading term of $X_2$ will be *minus;* and all these leading terms must be plus when the roots are all real (p. 95.)

If we conceive the coefficient $a$ to be introduced before $x^3$, then for $c$ we must write $\frac{c}{a}$ and for $b^2$, $\frac{b^2}{a^2}$: so that, multiplying each member of the inequality by $a^2$, the condition will be $3ac > b^2$. Hence, taking into account only the first three terms of a cubic equation, *if three times the product of the extremes exceed the square of the mean, two roots must be imaginary.* It is scarcely necessary to add that two *may* be imaginary although this condition have not place; for $X_3$ may be *negative.* But it may be interesting to notice, as an immediate inference from the above, as well as from the rule of Descartes, that when the second term of a cubic is absent, and the third *positive*, two roots are necessarily imaginary.

2. When the terms of X are alternately positive and negative all the roots *may* be positive. If they are, the final terms $d$, $c$, $bc - 9d$ must be alternately negative and positive; because only under such circumstances can *three* variations be lost in proceeding from $x = 0$ to $x = \infty$. Hence when the roots are all positive, not only the rule of Descartes, but the following condition must also be complied with, viz. *the product of the two mean terms of the proposed cubic must be greater than nine times that of the extremes.* Under this condition, if the roots are not all positive, two must be imaginary, and the final uncomputed remainder must therefore be *minus*. If the condition fail, two roots will also be imaginary; since the final signs will not then be alternately negative and positive, so that three positive roots cannot exist, and negative roots are forbidden by the rule of Descartes.

By proceeding with an equation of a higher degree than the third, in a manner similar to that above, analogous conclusions might be drawn.

## SECTION III.

### *On the Solution of Numerical Equations of the Second, Third, and Fourth Degrees.*

(92.) Having shown in the preceding section how the number and situations of the real roots of a cubic or biquadratic equation may always be ascertained, we are now prepared to enter upon the actual development of these roots, figure after figure. The solution of a quadratic equation is usually effected by an independent process, having little or no analogy to the method of continuous approximation which we are about to explain, and which is applicable to equations of all degrees. It is of importance, however, to bring quadratics, as well as the higher equations, within the range of this general mode of development; for however easy and convenient the common rule for quadratics may be, when the coefficients of the equation to be solved by it are small numbers, yet when these coefficients are very large, and the roots are required to be actually exhibited, and not merely indicated by an algebraical form, that rule involves a great deal of numerical calculation; and must yield, in point both of brevity and simplicity, to the universal method of continuous approximation. We shall therefore appropriate a short chapter to the solution of this class of equations; wherein it will be seen that the entire operation by the new method is attended with little more trouble than what is implied in the common process for the extraction of the square root of the number which enters the ordinary quadratic formula. And indeed as this number is always of greater magnitude than any of the numbers employed in the method about to be explained, it will generally be found, when large coefficients enter, that this latter method demands even fewer figures and less time than the single operation referred to. This operation, in the old method, is however sometimes left unperformed; the extraction being *indicated* merely by the prefix of the radical sign. But the result, left in this form, furnishes but an incomplete solution to the equation; and requires to be further developed before it can become available in any case of actual calculation, The student therefore will bear in mind, when instituting a comparison between the process here recommended, and that to which he has hitherto been accustomed, that the roots, determined by the common formula, ought to be fully developed, as in this new method, and not left concealed under the symbol of extraction.

## CHAPTER I.

### ON THE SOLUTION OF QUADRATIC EQUATIONS BY CONTINUOUS APPROXIMATION.

(93.) THE character of the roots of an equation of the second degree may generally be at once inferred from the signs of its second and third terms. Thus if the equation be

$$ax^2 + bx + c = 0$$

and $c$ be negative, one root of the equation must be positive and the other negative (23). If $c$ be positive, the roots, when real, must both have the same sign (25), which sign will be the opposite to that of the second term (25). The character of the roots is thus ascertained without any trouble as soon as the reality of them is established. The criterion of this reality we already know to be $b^2 - 4ac > 0$ (*Algebra*, art. 110); for equal roots it is $b^2 - 4ac = 0$, and for imaginary roots $b^2 - 4ac < 0$.

It is only, therefore, when $c$ is positive that any examination of the equation previously to determining the first figure of a root is at all necessary. But in this case it is proper to ascertain the sign of the quantity $b^2 - 4ac$ for the purpose of discovering whether or not the roots have any real existence. Even in the common method of solving quadratics it would be well to make the same preliminary enquiry, as much labour would occasionally be saved by it.

(94.) In this same case of $c$ positive, it may also happen that the roots, though previously ascertained, by the examination just recommended, to be real, may yet lie so closely together as to cause every one of our consecutive substitutions for $x$ to give a positive result, as they would do if the roots were imaginary. But the consecutive numbers between which both the roots actually lie, in such a case, or the first figure of each, may be immediately discovered; for that first figure will obviously be that which is also the first figure of half the second coefficient divided by the first, taken with a changed sign. Hence we need not call in the aid of Sturm's theorem for any formal analysis of an equation of the second degree. We shall therefore proceed at once to show how the entire root may be developed after the first figure has been determined.

(95.) We shall resume our former notation, and shall write the general form of a quadratic equation thus

$$A_2x^2 + Ax = N \; . \; . \; . \; . \; [1]$$

the absolute number N being transposed to the right-hand side merely for convenience.

Let the first figure of a root of this equation be represented by $r$; then $r$ is such a number that if N be divided by

$$A_2r + A$$

the quotient, or rather the leading figure of that quotient, will be $r$.* Let the remaining figures of this root be $r'$, $r''$, $r'''$, &c. in succession; then, putting for brevity the symbol $x'$ for the whole of this remaining portion of the root, we have $x = r + x'$, or, which is the same thing, $x' + r = x$. Consequently

$$\begin{array}{rcl} Ax' + Ar & = & Ax \\ \underline{A_2x'^2 + 2A_2rx' + A_2r^2} & = & \underline{A_2x^2} \\ \therefore A_2x'^2 + A'x' + R & = & N \end{array}$$

---

* It would be more strictly accurate to say that $r$ is such a number at it has to $A_2r + A$ the relation of quotient to divisor, in reference to , taken as the dividend. The shorter form of expression employed in e text merely avoids the circumlocution, but should be received in this nse.

where $A_2$, $A'$, R and N, being the sums of the numbers under which they are respectively placed, are all known. Let now R be carried over to the other side of the equation; and for brevity put $N'$ for $N - R$; we shall then have the equation

$$A_2x'^2 + A'x' = N' \; . \; . \; . \; . \; [\text{II}]$$

the roots of which are obviously those of the original equation [I] diminished by $r$ (36.) Hence $r'$, the second figure of the root we are developing of the proposed equation, is the first figure of a root of this; and is consequently such a number that if $N'$ be divided by

$$A_2r' + A'$$

the quotient, or at least the first figure of it, will be $r'$. If therefore this divisor were known, $r'$, the second figure of the sought root, could be immediately obtained, But it is impossible that we can know the *whole* of this divisor without also knowing $r'$; seeing that $r'$ enters into the formation of it. A *part* of this divisor, however, we do know, viz., the term $A'$, since this portion of it is entirely independent of $r'$; and if we examine into the formation of $A'$ and reflect that $r'$, occupying a place lower in the arithmetical scale than $r$, must in all cases be less than $r$, we shall readily perceive that, in most instances $A'$ must form a very large portion of the entire divisor, and being known may therefore be advantageously employed as an approximate divisor, or a *trial-divisor*, for finding $r'$. The accuracy of this estimated value of $r'$ may then be readily tested by completing the divisor with it, and trying whether or not our estimation of $r'$ is correct. In thus anticipating the new figure $r'$ of the root $x$, by the help of an incomplete divisor, the student will at once recognize a close analogy to the common arithmetical process for extracting the square root, where each succeeding figure of the quotient is thus suggested by a trial-divisor formed from the preceding figures, and verified by the true divisor which the new figure is employed to complete.

Suppose $r'$ to be determined in this manner, by help of the trial-divisor A′. Let the remaining portion of the root, consisting of the figures $r''$, $r'''$, &c. be called $x''$; then $x''+r'=x'$, so that by substitution in [II] we have

$$\begin{array}{rl} & A'x''+A'r'=A'x' \\ & \underline{A_2'x''^2+2A_2'r'x''+A_2'r'^2}=\underline{A_2'x'^2} \\ \therefore & A_2'x''^2+A''x''+R' \quad = \quad N' \end{array}$$

or, by transposing R, and putting N″ for N′ — R,

$$A_2'x''^2+A''x''=N'' \quad . \; . \; . \; [\text{III}]$$

an equation whose roots are those of the original diminished by $r+r'$; and consequently having for one of these roots the remaining portion $r''+r'''+$ &c. of the original root to which we are approximating, and of which the first two figures $r$, $r'$ are now supposed to have been discovered. Hence the third figure of the root sought, $r''$, is the first of a root of [III]. This first figure is the quotient of N″ by the divisor

$$A_2'r''+A''$$

and is, as in the former steps, suggested by the trial divisor A″; which suggested figure, being then employed as above, or indeed as in the common operation for the square root, to complete the true divisor, is thus submitted to the proper test of its accuracy.

It is obvious that by continuing this process we are conducted step by step through a series of transformed equations such, that each successive figure of the root sought is furnished by the *leading* figure of that in the transformed equation; each leading figure being suggested by the *previously computed* trial-divisor. And thus the required root may be accurately determined when consisting of a finite number of figures, or may be approximated to, to any extent, when an interminable decimal.

(96.) The transformed equations [I], [II], [III], &c. to which we are conducted step after step in this method of evolving the root, are very commodiously derived, one from the other, by the short process recommended at (37). In the above general investigation we have employed the older method of transformation instead; because we have thought it to be better adapted, in this particular enquiry, to clearness of exposition. But in the actual work of determining these transformed equations in practice, the more rapid and easy method of proceeding, already adverted to, will of course be preferred. And as in that method the coefficients only are retained, it follows that the development of the root, according to the foregoing instructions, is an operation of a character purely arithmetical, and unencumbered by any algebraical symbols.

No formal rule will be necessary to describe the steps of the process; for these steps are explained with sufficient detail at (37) we have only to remember here, in addition to the precepts there taught, that every figure of the root after the first, and by which the roots of the transformed equations are successively diminished, is suggested by a trial-divisor, previously computed, this trial-divisor being indeed no other than the second coefficient in the transformed equation itself.

(97.) It may however be proper to observe, before actually proceeding to examples, that we shall never be in any danger of mistaking the root, whose first figure we want, in any of these transformations. The student might at first sight suspect the possibility of such a mistake; because, whichever of the two roots of any transformed equation we refer to, the first figure of it will, of course, be such that, when introduced, as above, into the true divisor, it will furnish that same figure for the quotient; so that there are always *two* figures which *apparently* present equal claim to a place in the quotient. It might easily be shown however that the claims are not equal; but that the root, whose figures follow in order those already developed, will be that whose first figure is suggested by the trial-divisor; into which divisor, it is to be observed, those previously determined figures enter. But it will be sufficient to remind the learner that such a passage from one root to another would betray itself by a change of sign in the absolute number, which is of course to be avoided throughout the operation.

(98.) We shall now proceed to examples.

### EXAMPLES.

1. Required the roots of the equation

$$x^2 + 24290x - 74389791 = 0$$

The positive root of this equation is readily found to lie between 2000 and 3000; that is, the first figure of it is 2 in the place of thousands. Hence, proceeding as at (37)*, we have

| | | | | |
|---|---|---|---|---|
| | 1 | 24290 | 74389791 | $(2000 = r$ |
| | | 2000 | 52580000 | |
| True divisor | | 26290 | | |
| | | 2000 | | |
| Trial-divisor | | 28290 | 21809791 | $(700 = r'$ |
| | | 700 | 20293000 | |
| True divisor | | 28990 | | |
| | | 700 | | |
| Trial-divisor | | 29690 | 1516791 | $(50 = r''$ |
| | | 50 | 1487000 | |
| True divisor | | 29740 | | |
| | | 50 | | |
| Trial-divisor | | 29790 | 29791 | $(1 = r'''$ |
| | | 1 | 29791 | |
| True divisor | | 29791 | | |

---

* As in the foregoing general investigation, the absolute number is ransposed to the right; and consequently appears with changed sign, as lso in all the subsequent transformations. Hence, in deducing these, he successive absolute numbers arise from *subtracting* instead of from

Consequently the complete root is 2751. And the operation when more compactly arranged stands thus:

| | | | |
|---|---|---|---|
| 1 | 24290 | 74389791 | (2751 |
| | 2000 | 52580000 | |
| | 26290 | 21809791 | |
| | 2000 | 20293000 | |
| | 28290 | 1516791 | |
| | 700 | 1487000 | |
| | 28990 | 29791 | |
| | 700 | 29791 | |
| | 29690 | | |
| | 50 | | |
| | 29740 | | |
| | 50 | | |
| | 29790 | | |
| | 1 | | |
| | 29791 | | |

The other root of the quadratic is found by adding this to the coefficient of the second term of the proposed equation and changing the sign of the result. That root is therefore —27041.

In the preceding work the thick lines mark the places of the true divisors, and the thin ones those of the trial-divisors.

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adding as at (37). This modification is made merely for the purpose of assimilating the operation more closely to that of common division, and the extraction of the square root; which latter operation is only a particular case, implied in the more general problem above for solving any quadratic whatever, as will be more distinctly seen towards the close of the present chapter.

2. Required the roots of the equation

$$3x^2 + 5185\,x - 370734 = 0$$

The positive root of this equation is found to lie between 60 and 70; so that the first figure is 6 in the tens' place

| | | |
|---|---|---|
| 3 | 5185 | 370734 (68·765 |
| | 180 | 321900 |
| | 5365 | 48834 |
| | 180 | 44552 |
| | 5545 | 428200 |
| | 24 | 391657 |
| | 5569 | 3654300 |
| | 24 | 3358428 |
| | 5593 | 29587200 |
| | 21 | 27987875 |
| | 55951 | 1599325 (285717 |
| | 21 | 1119518 |
| | 55972 | 479807 |
| | 18 | 447807 |
| | 559738 | 32000 |
| | 18 | 27988 |
| | 559756 | 4012 |
| | 15 | 3918 |
| | 5597575 | 94 |
| | 15 | 56 |
| | 5597590) | 38 |
| | | 39 |

The figures 285717 are obtained by contracted division from the trial-divisor 5597590, which is curtailed of a figure at every step*.

---

* The student will not forget that the figure thus cut off at every step

The figures of the divisor thus employed would obviously be entirely unaffected by the subsequent additions to it, as new root figures are determined; so that the suppression of all these additions does not disturb the accuracy of our quotient within the limits to which the contracted method enables us to carry it. Hence the root is 68·765285717, which must be true to the nearest decimal. The other root is therefore — 1797·09861905.

It has been thought unnecessary to insert the decimal points in the body of the work after the integral part of the root has been found; for in extending the first column of the operation beyond this stage, the addition to it, which every new figure furnishes, uniformly advances one place to the right; so that the decimal point is not needed to regulate the arrangement. In the other column any such guide to the correct arrangement is still more unnecessary.

It is plain that after the integral part of the root is determined we may check the advance of the first or divisor-column as soon as we please; determining the trial-divisor at which we shall stop by the extent of decimals we propose to compute the root to. For the trial-divisor at which our contracted division commences will always furnish as many additional figures *minus* one as there are figures in that divisor itself*.

In the above development we have extended the root to nine places of decimals; six of which have been obtained by contracted division, as the trial-divisor, at which we stopped the advance of the divisor-column, contained seven figures. But it is seldom that so many as nine places of decimals are required: about half that number is amply sufficient for most practical purposes. Limiting our development then to five decimals we should only proceed

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in contracted division is not entirely disregarded: for, in multiplying the contracted divisor by the quotient figure, the amount *carried* from the rejected figure still goes to increase the product of the figures preserved, and to this amount a unit is added provided the number that would have been put down from the rejected figure be so great as *5*.

[*] Before we close the present chapter we shall point out a general method of contracting each column of the work, as the development proceeds, so as to bring no figures into operation except those which are really effective in determining the root to the desired number of places.

with the divisor-column till we arrive at a trial-divisor of six places of figures; and the whole work would be as follows:

| | | |
|---|---|---|
| 3 | 5185 | 370734 (68·76528 |
| | 180 | 321900 |
| | 5365 | 48834 |
| | 180 | 44552 |
| | 5545 | 428200 |
| | 24 | 391657 |
| | 5569 | 36543 |
| | 24 | 33583 |
| | 5593 | 2960 |
| | 21 | 2799 |
| | 55951 | 161 |
| | 21 | 112 |
| | 55972) | 49 |
| | | 44 |

It should be observed that the *last* decimal determined in this way is not always to be depended upon as strictly true to the nearest unit; although such will usually be the case. The instances in which we may doubt the strict accuracy of this final decimal are those in which the number that would have been placed under the divisor at which we stop, had the work of the divisor-column been carried on, has a considerable effect upon the magnitude of the first rejected figure in that divisor. In the example before us this effect is but inconsiderable; the first rejected figure, 2, in the contracted divisor above, is not increased by so much as two units by the 18 which completes the divisor.

It is plain, when the coefficient of the second term of the proposed equation, that is, the number at the head of the divisor-column, does itself involve several decimals, that unless the root sought be very minute, the numbers successively added to the divisor-column, as the figures of the root arise, will leave the final decimals at the head of the column undisturbed till after several steps of the work; that is, till a decimal in the root has been reached sufficiently remote to affect the divisor-column in the final decimals referred to.

We may therefore, in a case like this, commence our contracted divisions at the very outset of the work; stopping the advance of the divisor-column as soon as we have reached a decimal in the root that influences the last of those which our contractions have left. A single example of this will however better explain the character of the operation.

3. Required the positive root of the equation

$$x^2 + 1{\cdot}41421356x - 1{\cdot}73205081 = 0$$

The place of this root is evidently between 0 and 1; but nearer the latter. A trial or two proves it to lie between ·7 and ·8; the development is therefore as follows:

| | | | |
|---|---|---|---|
| 1 | 1·41421356 | 1·73205081 | (·78689818 |
| | 7 | 1·47994949 | |
| | 2·1142135\|6 | 25210132 | |
| | 7 | 23153708 | |
| | 2·8142135 | 2056424 | |
| | 8 | 1788128 | |
| | 2·894213\|5 | 268296 | |
| | 8 | 238961 | |
| | 2·974213 | 29335 | |
| | 6 | 26890 | |
| | 2·98021\|3 | 2445 | |
| | 6 | 2390 | |
| | 2·98621 | 55 | |
| | 8 | 30 | |
| | 2·9870\|1 | 25 | |
| | 8 | 24 | |
| | 2\|9\|8\|7\|8 | 1 | |

**4.** Required the positive root of the equation

$$3x^2 + 22x - 102 = 0$$

The first figure of the required root is 3.

| | | |
|---|---|---|
| 3 | 22 | 102 (3·2213261 |
| | 9 | 93 |
| | 31 | 900 |
| | 9 | 812 |
| | 40 | 8800 |
| | 6 | 8252 |
| | 406 | 54800 |
| | 6 | 41323 |
| | 412 | 13477 |
| | 6 | 12398 |
| | 4126 | 1079 |
| | 6 | 826 |
| | 4132 | 253 |
| | 3 | 248 |
| | 41323 | 5 |
| | 3 | 4 |
| | 41326) | 1 |

The root here developed is that of the quadratic $X_1 = 0$, which is found in the analysis of the cubic equation at page 96, to lie between 3 and 4. The root of $X_2 = 0$, the expression next following in that analysis, is obtained by simple division; and we have seen at (85) that by developing both these roots up to the point at which they separate we may complete the analysis of the proposed cubic without computing the final remainder, which final remainder we have calculated at page 113. Performing the division adverted to, we have

$$\begin{array}{l}
854)2751(3{\cdot}22131 \\
\quad 2562 \\
\hline
\quad\;\; 1890 \\
\quad\;\; 1708 \\
\hline
\quad\;\;\; 1820 \\
\quad\;\;\; 1708 \\
\hline
\quad\;\;\;\; 1120 \\
\quad\;\;\;\;\;\; 854 \\
\hline
\quad\;\;\;\;\; 2660 \\
\quad\;\;\;\;\; 2562 \\
\hline
\quad\;\;\;\;\;\;\; 980
\end{array}$$

Thus the expression $X_2$ reaches zero, and consequently changes sign first, in the passage of $x$ over the interval [3, 4]. Hence, referring to the analysis just quoted, we find the state of the signs immediately before and after the passage of the root of $X_1 = 0$, which lies in the proposed interval, to be (73)

$$\begin{array}{ccc}
X_1 & X_2 & X_3 \\
- & - & \\
 & & + \\
- & + &
\end{array}$$

showing that one variation is lost in the signs of $X_2$, $X_3$, in passing over the interval [3, 4]; and as one is also lost in the former part of the series, that is, in $X$, $X_1$, $X_2$, during the same passage, it follows that *two* roots of the proposed equation lie between 3 and 4.

The work here exhibited supplies the place of that part of the operation at page 113, which follows the close of the first step, or determination of 854 — 2751. It occupies considerably more space than that, because every number, aiding in the calculation, is actually put down; whereas in the multiplications at p. 113 the results only are exhibited.

It is probable that in this particular example the expenditure of time and trouble is about the same in both methods; as the numbers, entering the multiplications referred to, are not remarkably large; and it so happens that the roots developed above are very close together.

(99.) It may not be altogether out of place to observe here that in the foregoing operation of common division we have followed the unnecessarily lengthy course usually taken by beginners. There is however an abridged method, less practised by computers than it deserves to be, which greatly reduces the number of figures; and interferes little, if at all, with the rapidity of the performance.

In this method the *remainders* only are inserted in the work, the intermediate subtractive quantities being omitted; so that the work of the division above would be reduced to the following:

854)2751 (3·22131
1890
1820
1120
2660
980

To show with what expedition these remainders are derived, we shall detail the above operation in words; and for greater distinctness, shall express the figures of the remainders in *italics*.

Three times 4 are 12, and *nine* are 21; carry 2. Three times 5 are 15, and 2 are 17, and *eight* are 25; carry 2. Three times 8 are 24, and 2 are 26 and *one* are 27. Again: twice 4 are 8 and *two* are 10; carry 1. Twice 5 are 10 and 1 are 11, and *eight* are 19; carry 1. Twice 8 are 16 and 1 are 17 and *one* are 18. And so on to the end.

It is plain that by always dividing in this manner, the second column in the operation of developing a root of a quadratic equation, may always be considerably reduced in extent, and a saving of time also be effected. But, as the old method is that to which students are in general better accustomed, we shall not innovate upon it in the present examples. In the first, or divisor-column of the work, however, there are needless repetitions that ought certainly to be cancelled; for every trial-divisor is found by placing under the true divisor the number already standing over it, and then adding. It is plain that the addition may be as well performed without their repetition as with it; and thus the column may be advantageously contracted in extent, as in the xample next following.

5. Required the roots of the equation

$$804x^2 - 35109x + 278740 = 0$$

Here both roots are positive, since the second term is negative (93). It is easily seen that one lies between 10 and 20; and consequently the other between 30 and 40. We shall develope this latter as we shall have occasion for it hereafter; and, as recommended above, shall determine every trial-divisor by adding the true divisor to the number above it.

```
804     — 35109        — 278740 (33·237027548036
          24120        — 329670
          ------         ------
        — 10989          50930 *
          ------         46629
          13131          ------
           2412           430100
          ------          362316
          15543           -------
          ------           6778400
          17955            5490216
            1608           ---------
          -------          128818400
          181158           128313276
          -------          ---------
          182766               5051240000
             2412              3667222416
          --------             ----------
          1830072              13840175|84
          --------             12835290
          1832484              ----------
              5628              1004885
          --------               916806
          18330468               -------
          ---------               88079
          18336096                73344
                1608              ------
          -----------              14735
          1833611208               14669
          -----------              -----
          18336128|16                 66
                                      55
                                      --
                                      11
                                      11
                                      --
```

---

* The change of sign here merely shows that a root is passed over in the interval [0, 30], the root, namely, that was previously discovered to lie between 10 and 20.

Hence one of the roots of the proposed equation is

$$x = 33{\cdot}237027548036$$

the other is found by adding this to $-\dfrac{35109}{804}$ and changing the sign of the result.

6. Required a root of the equation.

$$46960x^2 - 307668x + 503794 = 0$$

The roots here are both positive; and moreover are near together, both commencing with the same figure (94). Consequently the first figure of $\dfrac{307668}{46960}$ will also be the first figure of each root (94); this first figure is therefore 3.

```
                                                  *
46960      — 307668            — 503794 (3·22059492
             140880            — 500364
           ─────────           ─────────
           — 166788             — 343000
           ─────────            — 330320
            — 25908             ──────────
              93920              — 1268000
           ─────────             — 1236960
           — 165160             ────────────
           ─────────              — 310400000
            — 71240               — 261106000
              93920              ─────────────
           ─────────               — 4930400000
           — 618480                — 4674972240
           ─────────               ─────────────
           — 524560                — 2554|27760
             234800                 — 2076
           ───────────              ──────
           — 52221200                — 478
           ───────────               — 467
           — 51986400                ─────
              422640                  — 11
           ────────────               — 10
           — 519441360                 ─
           ────────────
          — 519|018720)
```

---

* If instead of this number ·2, the figure ·3 had been assumed, the resulting true divisor would have been —118200; which, multiplied by the ·3, would have given —354600, which subtracted from the former remainder, —343000, would have produced a change of sign in the absolute number; thus apprizing us that, in proceeding from 3 to 3·3, a root is

(100.) The student may now form some estimate of the comparative advantages, on the grounds of simplicity and expedition, of the old and new methods of solving quadratic equations involving large coefficients. All the examples here solved are of considerable difficulty; and could not be worked by the common method without a much larger array of figures, and a proportionate increase of numerical calculation. It ought however to be noticed that the development of a root, by the preceding method, is not in every case to be accomplished with equal rapidity; for when the two roots lie near to each other, as in the example just solved, more than ordinary caution is requisite till we arrive at the point at which the roots separate; and we ought always to keep up a search for this point of separation, as our development proceeds, by trying the effect of adjacent figures on the *sign* of the absolute number, as briefly exemplified in the last foot-note. We recommend caution in cases of this kind, because there is more danger of passing unconsciously over *both* roots, and thus going, for a step, out of the direct line of development, than when the roots are wider apart. A change of sign in the first column will, however, always accompany this circumstance, since *both* roots cannot be diminished by more than their value without producing a change of sign in their sum.

(101.) It is proper further to remark that it has not been our object in the foregoing solutions to economise our figures to the utmost. We have, on the contrary, disregarded some obvious abbreviating contrivances that would have considerably reduced their number. In the last example, for instance, the right-hand figures 018720 in the final divisor have been rejected as useless, after the trouble of computing them. It is plain that, by a little

---

passed over. But as the figure ·2 produces no change of sign, we infer that the two roots separate at the second figure, having only the integral part 3 in common. It is worthy of remark too, that if ·4 be assumed as the second figure, the sign of the absolute number again changes; thus showing that the other root lies between 3·3 and 3·4; or that its first two figures are 3·3. We might therefore have preserved the ·3, and have thence proceeded with the development of the other root. We shall hereafter propose this development as an exercise for the student.

management, we might have so narrowed the divisor column after the first two or three steps, by disregarding the right-hand figures of 46960 one after another, as in the contracted division, as to have precluded the entrance of these waste figures; and thus have saved both space and labour.

Something of this kind has been already exhibited in the solution to example 3. But to exemplify more fully the general character of these abbreviations, we shall here present the foregoing operation curtailed of the redundant figures. In the examples hereafter given for exercise, the student may practise these reductions or not, as he pleases; but in long developments they certainly effect a considerable saving of time and space.

| | | |
|---|---|---|
| 46960 | —307668 | —503794 ( 3·220594921 |
| | 140880 | 500364 |
| | —166788 | —343000 |
| | —25908 | —330320 |
| | 93920 | —1268000 |
| | —165160 | —1236960 |
| | —71240 | —3104000 |
| | 93920 | —2611060 |
| | —618480 | —493040 |
| | —524560 | —467498 |
| 469\|6 . . . . | 2348 | —25542 |
| | —522212 | —20760 |
| | —519864 | —4782 |
| 46\|9 . . . . | 422 | —4671 |
| | —51944\|2 | —111 |
| | —51902 | —104 |
| 4\|6 . . . . | 2 | —7 |
| | —5190\|0 | —5 |

We have exhibited the number 46960, with its successive contractions, opposite the places where it is used; but this repetition is not necessary in practice: the rejected figures may be crossed off with the pen as they fall into disuse.

(102.) We shall now propose a few examples by way of exercise in the preceding method; and, for the better accommodation of the student, shall here state the essentials of the operation in a *rule;* thus:

Rule i. Find by trial the leading figure of a root.

ii. Multiply this by the first coefficient, and add the product to the second coefficient: the result is the first *true divisor;* the absolute number forming the dividend, and the root-figure the commencement of the quotient.

iii. To the true divisor add the number immediately over it; this will give the *trial-divisor* for determining the next root-figure, from which form the true divisor as before, and then proceed as at first.

7. Required the roots of the equation

$$x^2+x-60=0$$

Ans. 7·262087348 and
— 8·262087348.

8. Required the roots of the equation

$$x^2-13{\cdot}97196x+48{\cdot}63627$$

Ans. 6·57653
and 7·39543.

9. Required the positive root of the equation

$$x^2+765432x-123456=0$$

to eight places of figures.

Ans. ·16128927.

10. Required the greater root of the equation

$$49x^2 - 1316x + 4137 = 0$$

to six places of figures.

Ans. 23·2205.

11. Required the roots of the equation

$$32x^2 - 220x + 897 = 0$$

Ans. The roots are imaginary.

12. Develope the negative root of the equation

$$17x^2 - 23x - 45 = 0$$

to ten or eleven places of decimals.

Ans. — 1·08553715303.

13. Required the two roots of the equation

$$12716x^2 + 2023x - 1695 = 0$$

to six or seven places of decimals.

Ans. ·2941176 and
— ·4532085.

14. Develope the positive root of the equation

$$17x^2 - 23x - 45 = 0$$

to ten or eleven places of decimals.

Ans. 2·43847832950.

15. Required the least root of the equation

$$3073059x^2 - 20133742x + 32968246 = 0$$

to eight places of decimals.

Ans. 3·22088567.

16. One root of the equation

$$46960x^2 - 307668x + 503794 = 0$$

developed at page 157 to eight places of decimals: required the velopment of the other root to the same extent.

(103.) It may be interesting to notice that if both roots of a quadratic equation be developed as in the preceding examples, the two series of divisors, disregarding their signs, will approximate nearer and nearer to one another as the operation proceeds; the leading figures may be made to agree, by extending the work, to any assigned number of places; and the final divisors will become identical when the roots themselves are not interminable decimals. This may be proved as follows:

Let $a$, $a_2$ be the roots of the quadratic equation

$$A_2x^2 + Ax + N = 0$$

then, when the root $a$ is developed by the above operation, the last divisor will be the number which occurs under A, in the result which arises from diminishing the roots of the equation by $a$: this result is obtained by our usual method, thus,

$$\begin{array}{llll}
 & A_2 & A & (\,a \\
 & & aA_2 & \\ \cline{3-3}
 & & aA_2 + A & \\
 & & aA_2 & \\ \cline{3-3}
\text{final divisor} & & 2aA_2 + A & \\ \cline{3-3}
\end{array}$$

Again: $a_2$, being the other root of the equation, is equal to $-\left(\frac{A}{A_2} + a\right)$; so that finding, in the same manner, the divisor for this root, we have

$$\begin{array}{llll}
 & A_2 & A & \left(-\frac{A}{A^2} - a\right. \\
 & & -A - aA_2 & \\ \cline{3-3}
 & & -aA_2 & \\
 & & -A - aA_2 & \\ \cline{3-3}
\text{final divisor} & & -A - 2aA_2 & \\ \cline{3-3}
\end{array}$$

which, disregarding the sign, is the same as before. And we obviously approach nearer and nearer to this identity as we approach nearer and nearer to the values of $a$ and $a_2$.

(104.) It is also worthy of remark that the foregoing operation for the development of a root of a quadratic equation comprehends in it, as a particular case, the common process for extracting the square root of a number; and that the general investigation at (95) supplies the rule for that process as an obvious corollary from the general conclusion.* Considerable analogy must already have been observed by the student between the rule referred to and the more general method by which a complete quadratic equation is solved. The analogy might have been rendered much closer if we had always prepared our equation for solution by depriving it of the leading coefficient, or if we had chosen only equations so prepared.

For instance, the work of the equation

$$x^2 + 24290x - 74389791 = 0$$

solved at page 148, might have been carried on as below, where the second coefficient is written over the place destined to receive the quotient figures as they become known; which figures are written severally under those above them that have the same denomination in the arithmetical scale; that is, thousands under thousands, hundreds under hundreds, tens under tens, and so on. This arrangement being adopted at the outset, it is easy to see that the first true divisor is the root figure, found by trial, incorporated with the number (viz. A) written above it. And every subsequent trial-divisor is got by incorporating *twice* the number already in the root with the number above; which trial-divisor, corrected by the addition to it of the new figure found by it, is the true divisor.

---

* This circumstance will not be regarded as altogether unimportant by the scrupulous student, because the ordinary algebraic demonstrations of .e common rules for extracting the square and cube roots of numbers e by no means remarkable for their perspicuity and conclusiveness. For me judicious observations on this subject the student is referred to the rticle on the square root in Davies's "Solutions to the Questions in utton's Mathematics."

**Note.** The best way of doubling the number already in the root, and incorporating this double with the number above, to get the trial-divisor, is simply to add the last found figure to the preceding true divisor; just as the next true divisor is found from this trial divisor. This is in accordance with the most approved method of extracting the square root,[*] and is exactly the way in which the divisor column is prolonged in all the foregoing examples.

The operation conducted upon this plan may be carried on with great ease and rapidity, as a few moments' contemplation of the work will show. The first is the only true divisor not actually exhibited. When the leading figure 2, under the 4, is found, we multiply the 24290 by this 2, till we come to the 4 placed over it; the 2 is then incorporated with the 4, which becomes 6, and the multiplication is continued, producing the number 52580.

| | | |
|---|---|---|
| | | 24290 |
| | 74389791 | ( 2751[†] = one root |
| | 52580000 | |
| 28290 | 21809791 | − 27041 = the other root |
| 28990 | 20293000 | |
| 29690 | 1516791 | |
| 29740 | 1487000 | |
| 29790 | 29791 | |
| 29791 | 29791 | |

---

[*] See Hutton's Course, by Davies; vol. i. page 67. 1841.

[†] It is obvious from (25), that when one root of a quadratic is thus determined, we have only to add it to the number above it, and to change the sign of the result, to obtain the other root.

If our quadratic equation were deprived of its second term, that is, if the number placed over the root figures were zero, the operation would then become that of extracting the square root of a number. For instance, if the example were

$$x^2 - 7568001 = 0$$

then, conforming exactly to the preceding directions, we should develope $x$ as follows

| | | |
|---|---|---|
| | 7568001 | ( 2751 |
| | 4000000 | |
| 4000 | 3568001 | |
| 4700 | 3290000 | |
| 5400 | 278001 | |
| 5450 | 272500 | |
| 5500 | 5501 | |
| 5501 | 5501 | |

And this becomes identical with the common operation for the square root when the superfluous ciphers, and the unnecessary repetitions in the divisors, are suppressed.

The student is recommended to work examples 7, 8, and 9 in conformity to the foregoing arrangement, which not only economises the figures, but also has the advantage of bringing the several divisors and their corresponding dividends into juxtaposition.

## CHAPTER II.

### ON THE SOLUTION OF CUBIC EQUATIONS.

(105.) THE investigations and developments of the preceding chapter, on the solution of quadratic equations by continuous approximation, furnish an easy and effective preparation for the discussion of equations of the third degree; and readily suggest the proper course to be pursued in the solution of numerical equations in general.

Let $r$ be the first figure of one of the roots of the cubic equation

$$A_3x^3 + A_2x^2 + Ax = N \quad . \; . \; . \quad [1]$$

then $r$ must be such that if N be divided by

$$A_3r^2 + A_2r + A$$

the quotient, or rather the first figure of the quotient will be $r$; or, to be still more explicit, the expression just written must bear to the figure $r$ the relation of divisor and quotient in reference to the dividend N. Let the remaining part of the root be called $x'$; and let it consist of the figures $r'$, $r''$ $r'''$, &c. Then, since $x = r + x'$, or which is the same thing $x' + r = x$, we have

$$\begin{array}{rcl} Ax' + Ar & = & Ax \\ A_2x'^2 + 2A_2rx' + A_2r^2 & = & A_2x^2 \\ A_3x'^3 + 3A_3rx'^2 + 3A_3r^2x' + A_3r^3 & = & A_3x^3 \\ \hline A_3x'^3 + A'_2x'^2 + A'x' + R \qquad & = & N \end{array}$$

where $A_3$, $A'_2$, $A'$, R, and N are the sums of the quantities under which they are placed, and are consequently known numbers.

By transposing R, and then putting N′ for N — R, we thus have the transformed equation

$$A_3x'^3 + A'_2x'^2 + A'x' = N' \ . \ . \ . \ . \ [\text{II}]$$

whose roots are those of [I] diminished by $r$; so that the first figure of one of the roots of this equation must be $r'$, the *second* figure of the root of [I] to which we are approximating. This $r'$ is related to the expression

$$A_3r'^2 + A'_2r' + A'$$

as quotient to divisor in reference to the dividend N′; and would therefore become known if this divisor were known. But a portion of it only is known, viz. the term A′, independent of $r'$, which however from its formation, and from the circumstance that $r$ is always a grade higher in the arithmetical scale than $r'$, must in general constitute an important part of the whole divisor, and may therefore be employed as an approximate or trial-divisor to suggest the quotient figure $r'$; which figure may then be easily tested by completing the divisor with it and actually performing the division.

The figure $r'$ being discovered and verified in this manner, and the remainder of the root of [II] to which we are approximating being called $x''$, and of which the first figure is by hypothesis $r''$, we shall have a new transformed equation in $x''$, by substituting in [II] $x'' + r'$ for $x'$; that is by diminishing the roots of [II] by the figure $r'$. As before, the form of this equation will be

$$A_3x''^3 + A''_2x''^2 + A''x'' = N'' \ . \ . \ . \ . \ [\text{III}]$$

and the leading figure of that root of this which we are thus following through all these transformations being $r''$, the expression

$$A_3r''^2 + A''_2r'' + A''$$

must bear towards $r''$ the relation of divisor and quotient in reference to the dividend N″. And for reasons similar to those advanced in the preceding step, this quotient will be suggested by the approximate or trial-divisor A″ which is already known.

Thus the several figures $r$, $r'$, $r''$, $r'''$, &c. of the root we are desirous of obtaining are determined one after another in regular succession, by descending through as many transformed equations, so contrived that each figure in succession necessarily becomes the *leading* figure of a root in the corresponding transformed equation. To pass from one of these transformations to the next in order we see that it is only necessary to know the *leading figure* in the former; this knowledge obtained, the passage may be effected with ease and expedition by the method explained at (37), and so often employed in last chapter. The problem, therefore, of completely developing the several figures of the root of an equation—for it is easy to see that the method is *general*—is thus reduced to the determination of the *leading* figure merely of a root in each of a dependent series of transformed equations. And the distinguishing peculiarity of the process, and that indeed which gives it nearly all its practical value, is the circumstance that, after the initial figure of the root is found by the infallible method of Sturm, or in any other way, and a commencement thus furnished for our series of transformations, *each transformed equation suggests its own leading figure.* The compact and purely arithmetical and continuous character which is given to the entire operation, by the manner in which the several transformations are effected, is no doubt an important feature in this method; and contributes much to practical facility in the numerical work. But the cardinal merit of Horner's discovery consists in the rapidity with which the several *trial-divisors* may be formed, their efficiency in discovering the successive leading figures, and the ease with which the true divisors may afterwards be completed.

## EXAMPLES.

(106.) 1. Required a root of the cubic equation.

$$x^3 + 2x^2 + 3x - 13089030 = 0$$

From the sign of the last term of this equation it has at least one positive root (22); and therefore, by the rule of Descartes, the other roots if real, must be both negative. The situation of the positive root is determinable without the aid of Sturm's theorem; since those substitutions for $x$ which inclose this root must of necessity cause the above polynomial to furnish results with opposite signs, so that we have only to seek for two consecutive numbers that give such results. These are readily found to be 200 and 300; consequently 2 is the first figure of the positive root, which 2 is in the hundreds' place. The process of development is therefore as follows, the successive transformations being effected as already described at (37), and as employed in last chapter.

| $A_3$ | $A_2$ | $A$ | $N$ |
|---|---|---|---|
| 1 | 2 | 3 | 13089030 ($200 = r$ |
| | 200 | 40400 | 8080600 |
| | 202 | 40403 | |
| | 200 | 80400 | |
| | 402 | | |
| | 200 | | |
| | $A'_2 = 602$ | $120803 = A'$ | 5008430 ($30 = r'$ |
| | 30 | 18960 | 4192890 |
| | 632 | 139763 | |
| | 30 | 19860 | |
| | 662 | | |
| | 30 | | |
| | $A''_2 = 692$ | $159623 = A''$ | 815540 ($5 = r''$ |
| | 5 | 3485 | 815540 |
| | 697 | 163108 | |

Hence the complete root is 235. And by adopting a more compact arrangement the above work will stand as follows, the thick lines, as before, marking the places of the true divisors:

| | | | | |
|---|---|---|---|---|
| | 1 | 2 | 3 | 13089030 (235 |
| | | 200 | 40400 | 8080600 |
| | | 202 | 40403 | 5008430 = $N'$ |
| | | 200 | 80400 | 4192890 |
| | | 402 | 120803 = $A'$ | 815540 = $N''$ |
| | | 200 | 18960 | 815540 |
| $A'_2 =$ | | 602 | 139763 | |
| | | 30 | 19860 | |
| | | 632 | 159623 = $A''$ | |
| | | 30 | 3485 | |
| | | 662 | 163108 | |
| | | 30 | | |
| $A''_2 =$ | | 692 | | |
| | | 5 | | |
| | | 697 | | |

We need not stop to point out the obvious contraction of which the first column of the work is susceptible, by the suppression of the repetitions, as in quadratic equations: the student may adopt these abridgments or not, as he thinks proper; we think however that, as a general principle, it is better to dispense with these contractions of the first column in all equations higher than a quadratic. A convenient abbreviated method is given in next chapter.

When, as in the present example, the root developed is integral and consequently accurately determinable, then the result of diminishing by this root must be a final transformed equation one of whose roots has been reduced to zero. This final transformed equation we have not fully worked out above, as the closing of the operation, by the exhaustion of N, renders the completion of the next following transformation useless for the purpose of further development.

To complete it requires that we carry on the first column two steps further, the second one step, and put in the third column 0 for $N'''$; we thus have the final numbers

$$1 \qquad 707 \qquad 166618 \qquad 0$$

and thence the final equation

$$x'''^3 + 707x'''^2 + 166618x''' = 0$$

one of whose roots is plainly $x''' = 0$, the other roots being involved in the quadratic equation

$$x'''^2 + 707\,x''' + 166618 = 0$$

so that the roots of this are the remaining roots of the proposed diminished by 235. These remaining roots then are imaginary; for $4 \times 166618 > 707^2$.

It is thus obvious that whenever the development of the root of a cubic equation is completed, that is whenever the absolute number is reduced to zero, the final transformation to which this zero belongs furnishes a depressed equation of the second degree whose roots, each increased by the root already developed, give the remaining roots of the proposed cubic.

When the root to be developed is an interminable decimal the absolute number can never be completely exhausted, although by extending the approximation it may be reduced to any degree of minuteness we please. Whenever in such a case we put a stop to the process, after having obtained any finite number of decimals, and regard this finite number as the complete root, we at the same time of course regard the small portion of N still unexhausted as unimportant, and reject it as if it were absolutely zero. Hence, as above, the results of the remaining columns, completing the final transformation to which this assumed zero belongs, furnish a quadratic equation, whose roots, increased by that already developed, are the remaining roots of the cubic equation, true to as many figures, or as many, wanting one, as the former root has been computed to. The following example will illustrate this.

2. Required a root of the equation

$$x^3 + 8x^2 + 6x - 75{\cdot}9$$

to about four places of decimals.

This equation like the last has a single positive root. It lies between 2 and 3

| | | | | |
|---|---|---|---|---|
| $1=A_3$ | $8=A_2$ | $6=A$ | 75·9 (2·4257 | $r=2$ |
| | 2 | 20 | 52 | |
| | 10 | 26 | 23·900 $N'$ | $r'=·4$ |
| | 2 | 24 | 22·304 | |
| | 12 | $50=A'$ | 1·596000$N''$ | $r''=·02$ |
| | 2 | 5·76 | 1·239688 | |
| | $14=A'_2$ | 55·76 | ·356312000$N'''$ | $r'''=·005$ |
| | ·4 | 5·92 | ·311827625 | |
| | 14·4 | $61·68=A''$ | 44484375000$N''''$ | $r''''=·0007$ |
| | ·4 | ·3044 | 43716797593 | |
| | 14·8 | 61·9844 | 767577407$N'''''$ | |
| | ·4 | ·3048 | | |
| | $15·2=A''_2$ | $62·2892=A'''$ | | |
| | 2 | 76325 | | |
| | 15·22 | 62·365525 | | |
| | 2 | 76350 | | |
| | 15·24 | $62·441875=A''''$ | | |
| | 2 | 1069299 | | |
| | $15·26=A'''_2$ | *62·45256799 | | |
| | 5 | 1069348 | | |
| | 15·265 | $62·46326147=A'''''$ | | |
| | 5 | | | |
| | 15·270 | | | |
| | 5 | | | |
| | $15·275=A''''_2$ | | | |
| | 7 | | | |
| | *15·2757 | | | |
| | 7 | | | |
| | 15·2764 | | | |
| | 7 | | | |
| | $15·2771=A'''''_2$ | | | |

---

* The development to the proposed extent of decimals terminates at this step; but the final transformation is completed in order to get the coefficients of the depressed quadratic equation.

Hence the quadratic equation furnishing approximate values of the remaining roots, each diminished by 2·4257, is

$$x^2+15{\cdot}2771x+62{\cdot}46326147=0$$

and these approximate values may be depended upon as far as four or five places of figures but not further.

It is plain that if by extending the foregoing development indefinitely we were actually to exhaust the absolute number, we should then literally obtain a quadratic furnishing the remaining roots. But the leading figures of the first two columns, supplying the coefficients of this quadratic would still be the same as at present; that is, we should have for the final results of these columns

$$15{\cdot}2771\ \ldots\ldots\ ;\ 62{\cdot}463\ \ldots\ldots\ldots$$

so that employing only these leading portions of the true coefficients and developing the roots of the quadratic by the method taught in last chapter we should succeed in discovering the true figures till the absolute number 62·463 became exhausted, or at least till it became reduced to a single figure; that is till our development reached the fourth or fifth figure. Thus our quadratic, curtailed of the inadmissible figures, is

$$x^2+15{\cdot}2771x+62{\cdot}463=0$$

But as four times the product of the extreme terms exceeds the square of the middle term, the roots of this quadratic are imaginary.

(107.) In order that the student may the more clearly perceive the strict conformity of the arithmetical operation above exhibited to the steps of the general investigation, we shall here write down the several transformed equations through which that operation has conducted us, and shall number them in the order of their occurrence, as in the theoretical investigation:

$$x^3 + 8x^2 + 6x = 75{\cdot}9 \quad \ldots\ldots \quad [\text{I}]$$

$$x'^3 + 14x'^2 + 50x' = 23{\cdot}9 \quad \ldots\ldots \quad [\text{II}]$$

$$x''^3 + 15{\cdot}2x''^2 + 61{\cdot}68x'' = 1{\cdot}596 \quad \ldots\ldots \quad [\text{III}]$$

$$x'''^3 + 15{\cdot}26x'''^2 + 62{\cdot}2892 = {\cdot}356312 \quad \ldots\ldots \quad [\text{IV}]$$

$$x''''^3 + 15{\cdot}275x''''^2 + 62{\cdot}441875x'''' = {\cdot}044484375 \quad \ldots\ldots \quad [\text{V}]$$

$$x^{v3} + 15{\cdot}2771x^{v2} + 62{\cdot}46326147x^{v} = {\cdot}000767577407 \quad \ldots\ldots \quad [\text{VI}]$$

The roots of [II] are those of [I] diminished by $r = 2$;
The roots of [III] are those of [I] diminished by $r + r' = 2{\cdot}4$;
The roots of [IV] are those of [I] diminished by $r + r' + r'' = 2{\cdot}42$;
The roots of [V] are those of [I] diminished by $r + r' + r'' + r''' = 2{\cdot}425$;
And finally the roots of [VI] are those of [I] diminished by $2{\cdot}4257$.

(108.) It is obvious, on account of the permanency of the leading figures 62·4 of the divisor last employed in the foregoing work, that two or three more correct decimals in the root might have been obtained by contracted division. And as the figures of the final divisor after these three are useless for this purpose, it would have been better to have provided against their entrance by gradually reducing the work from some preceding step, as in the operation for quadratics at page 159.

In ascertaining at what step of the process these reductions may commence, we should previously determine to what number of places of figures our root is to be carried. We should then proceed with the work till we arrive at that absolute number in the last column which contains as many places of figures, or as many plus one, as are still wanting in the root to complete our predetermined number. From that step our contractions are to commence; and it is obvious that if they be so managed as to preclude the entrance of figures into the last column, beyond the boundary which limits the before-mentioned absolute number, we shall, as in contracted division, secure accuracy for our quotient figures till this absolute number becomes reduced to a single figure or is completely exhausted; and this it may be, under peculiar relations of the coefficients, before the ordinary number of root-figures is reached.

(109.) We shall illustrate the nature of these contractions by means of the preceding example, where it is required to carry the root to four decimals, or to five places of figures. It is seen, after two places in the root are found, that our resulting absolute number, 1596, consists of four places; and, as only three more decimals in the root are to be obtained, we check the advance of the last column beyond the limit already reached. The next divisor must therefore recede back a figure; the next another figure; and so on. The numbers in the first column, which contribute to these divisors, instead of advancing figure by figure, as in the unabbreviated operation, must now, on the contrary, recede figure by figure; that an over-supply may not be carried from them to the divisors.

These successive retrenchments speedily exhaust this first column; and thus naturally reduce the operation to one of common contracted division.

It must be remembered that in all these contractions the figures *carried* from rejected numbers are not to be neglected: the figures retained are to receive the full benefit of these carryings, as in contracted division; and, like as in this division when the number that would have been written down is so great as 5, the carrying is increased by a unit.

```
1       8         6        75·9 (2·4257
        2        20        52
       ---       --        ----
       10        26        23·900
        2        24        22·304
       ---       --        ------
       12        50         1·596
        2         5·76      1·240
       ---       -----      -----
       14        55·76       ·356
         ·4       5·92       ·312
       ----      -----       ----
       14·4      61·68        44
         ·4        ·30        43
       ----      -----        --
       14·8      61·9|8        1
         ·4        ·3
       ----      ----
      1|5|·2     62·3
                   ·1
                 ------
                 6|2|·4
```

3. Required the positive root of the equation

$$x^3+5x^2+7x-47=0$$

to four places of decimals.

The first figure of the positive root is 2; and the operation is as follows:

| | | | |
|---|---|---|---|
| 1 | 5 | 7 | 47 (2·1238 |
| | 2 | 14 | 42 |
| | 7 | 21 | 5·000 |
| | 2 | 18 | 4·011 |
| | 9 | 39 | ·989 |
| | 2 | 1·11 | ·829 |
| | 11 | 40·11 | ·160 |
| | ·1 | 1·12 | ·125 |
| | 11·1 | 41·23 | 35 |
| | ·1 | ·23 | 33 |
| | 11·2 | 41·4\|6 | 2 |
| | ·1 | ·2 | |
| | 1\|1\|·3 | 41·7 | |
| | | 0 | |
| | | 4\|1\|·7 | |

4. Required a positive root of the equation

$$2x^3-3x^2-7x+6=0$$

This equation has been shown at page 120 to have two positive roots: one between 0 and 1, and the other between 2 and 3. We shall proceed to develope the latter.

$$\begin{array}{rrrrl}
2 & -3 & -7 & -5 & (\ 2{\cdot}5 \\
 & \underline{4} & \underline{2} & \underline{-10} & \\
 & 1 & -5 & 5\text{[*]} & \\
 & \underline{4} & \underline{10} & \underline{\underline{5}} & \\
 & 5 & 5 & & \\
 & \underline{4} & \underline{5} & & \\
 & 9 & 10 & & \\
 & \underline{1} & & & \\
 & 10 & & &
\end{array}$$

Hence the root sought is 2·5.

By completing the final transformation, we have the depressed quadratic equation

$$2x^2 + 12x + 15{\cdot}5 = 0$$

or

$$x^2 + 6x + 7{\cdot}75 = 0$$

the roots of which we know from (30), as well as from (93), must be both negative. For convenience we shall convert them into positive, by changing the sign of $6x$; and shall then solve the equation after the manner explained at (104). The first figure of a root, after this change, is 1.

---

[*] The change of sign here shows that a root lies in the interval [0, 2].

```
                  —6
          *7·75 ( 1·88196605
            5
 4      |  275
 3·2    |  256
 ------------------
 2·4    |   1900
 2·32   |   1856
 ---------------------
 2·24   |     4400
 2·239  |     2239
 -----------------------
   2238  |  216100
   22371 |  201329
 -----------------------
 2|2|3|6|2)  14771
             13417
             -----
              1354
             ·1342
              ----
                12
                11
                --
                 1
                --
```

(110.) The other root of this quadratic equation is got by adding the root just determined to the —6 above it, and changing the sign of the result: this other root is therefore 4·11803395. The two roots thus found, when taken negatively, are those of the original equation diminished by 2·5. Hence, prefixing to them

---

* The minus signs belonging to the numbers in this column, as also those belonging to the several divisors, are omitted, as being unnecessary. It will be remembered that every trial-divisor is got by adding the last root-figure to, or in this case subtracting it from, the true divisor; like as the true are got from the trial-divisors. (See Note, page 164.)

the minus sign, and increasing each by this quantity, we have the three roots of the proposed equation as follows, viz.

$$2{\cdot}5\,;\quad {\cdot}61803395\,;\quad -1{\cdot}61803395$$

the sum of which is 1·5, the second coefficient of the proposed equation, taken with a changed sign, and divided by the first; as it ought to be.

(111.) The root 2·5, first determined, is that in the interval [2, 3], considered in example 1 at page 119. And the root lying in the same interval, of the subsequent quadratic $X_2 = 0$, has been found at page 161, ex. 14, to be 2·43.... Hence, in passing over this interval, $X_2$ arrives at zero before $X_1$; and thus two roots of the biquadratic are detected in the interval under examination, as at p. 120, since two variations are lost in the passage over it.

(112.) From what has been shown in the preceding example, and in example 2 at page 171, it is obvious that we can always develope the real roots of a cubic equation without submitting it to a preliminary analysis by the theorem of Sturm. For, as one root of such an equation is necessarily real (22), the situation of a real root may always be discovered by successive substitutions. If two roots should lie so closely together as not to be separable by these substitutions, then the place of the third root will be detected; and if all three thus approximate, then the interval they occupy will become known by the consecutive substitutions which comprehend it giving opposite signs. In all cases, therefore, we can arrive at a leading figure of a root of a cubic equation without the aid of Sturm's theorem, and can thus lay a sufficient foundation for the development of that root.

When one root is developed, either completely or to a sufficient extent of decimals, then, provided no abbreviations have been introduced into the operation, we shall have been led, as in the examples just referred to, to a depressed quadratic equation, from which the remaining roots may be derived. But if abbreviations have been employed, it is possible that the character of the depressed equation may be doubtful, for want of the suppressed figures; then the quadratic, involving the two remaining roots, is to be found by adding the root already determined to the coefficient of the second term of the cubic (the first coefficient being removed by division) for the coefficient of the second term of the sought quadratic; and by dividing the absolute term of the cubic by the same root, changing the sign of the result, for the absolute term of the quadratic (25).

In many cases either of these methods, indifferently, might be employed with advantage; but in examples of any delicacy, that is, in those instances where the criteria (93) furnished by the depressed quadratic would require a minute examination before it could be determined whether the roots of it are nearly equal or imaginary,—we say that in instances such as these, in which abridgments of the coefficients of our quadratic might alter the true character of its roots, the accuracy with which these coefficients must be determined and examined would entail upon the computer much more numerical labour than the previous analysis of the equation would in general require. In ordinary cases, however, we may carry on our abridgments in developing a root, and yet preserve sufficiently clear indications, in our abridged coefficients, of the true character of the remaining roots, and thence be enabled to determine the leading figures of those roots when real. But when their character is not thus clearly indicated by the abridged coefficients, we ought to have recourse to the theorem of Sturm to test them, and to supply their leading figures. The student must not, however, view this theorem exclusively in connexion with the practical development of the roots of equations; for the ultimate object in many analytical enquiries is attained when the character and situations merely of the roots of the resulting equation are discovered; although the precise arithmetical values of them remain undetermined.

(113.) The following example will tend to illustrate and confirm some of the preceding observations.

5. Required the roots of the equation

$$x^3 - 17x^2 + 54x - 350 = 0$$

Without the aid of Sturm's theorem it is easily discovered that one root of this equation lies between 10 and 20. But we cannot tell whether this interval comprehends only one root, or all three; or whether two of the roots are imaginary. As, however, one root at least lies in the interval [10, 20] let us proceed to develope it:

$$\begin{array}{rrrrl}
1 & -17 & 54 & 350 & (\,14 \\
 & \underline{10} & \underline{-70} & \underline{-160} & \\
 & -7 & -16 & 510 & \\
 & \underline{10} & \underline{30} & \underline{328} & \\
 & 3 & 14 & 182 & \\
 & \underline{10} & \underline{68} & & \\
 & 13 & 82 & & \\
 & \underline{4} & \underline{84} & & \\
 & 17 & 166 & & \\
 & \underline{4} & & & \\
 & 21 & & & \\
 & \underline{4} & & & \\
 & 25 & & &
\end{array}$$

Let us pause for a moment at this transformation, which has conducted us to the equation

$$x^3 + 25x^2 + 166x - 182 = 0$$

in which the changes of sign are fewer by two than those in the original equation. If the roots are real, those of the original must be all positive, by the rule of Descartes, whilst those of the transformed equation must be one positive and two negative. We infer, therefore, either that two real roots of the equation are passed over in proceeding to the above transformation—that is, that the diminution of the original positive roots by 14 converts two of them into negative roots—or else that two roots of the equation are imaginary. But we have no means of ascertaining, from what is now done, which of these two conclusions is the correct one. We must remain in uncertainty till we have deduced and examined the depressed quadratic equation.

Let us resort to the theorem of Sturm for information on this point.

From the original equation we have

$$X = x^3 - 17x^2 + 54x - 350$$

$$X_1 = 3x^2 - 34x + 54$$

the coefficients in which furnish, according to the rule (78), the two following rows

$$\begin{array}{rrr} 3 & -\ 34 & +\ 54 \\ -\ 17 & -\ 108 & +\ 1050 \end{array}$$

But these numbers may be reduced by the precepts at (77); for, omitting the multipliers, the remaining four may be written

$$\begin{array}{rr} -17 & +\ 27 \\ -54 & +\ 525 \end{array}$$

Hence the work is as follows:

$$\begin{array}{r|rr} 3 & -17 & +\ 27 \\ -17 & -54 & +\ 525 \\ \hline & 289 & -\ 459 \\ & -162 & +\ 1575 \\ \hline & 127 & +\ 1116 \end{array}$$

$$\therefore X_2 = 127x + 1116$$

$$\therefore X_3 = -$$

As the final sign is negative, we infer (p. 95) that two roots of the proposed equation are imaginary. We need scarcely add that this method of determining the character of the doubtful roots occupies but a few moments of time. The character of these roots would, however, have as clearly unfolded itself if we had continued the development. Thus, resuming the last transformed equation, the continuation of the work is as follows:

| | | | |
|---|---|---|---|
| 1 | 25 | 166 | 182·000 ( ·95406861 |
| | 9·9 | 23·31 | 170·379 |
| | 25·9 | 189·31 | 11·621000 |
| | ·9 | 24·12 | 10·740875 |
| | 26·8 | 213·43 | ·880125 |
| | ·9 | 1·3875 | ·865276 |
| | 27·7 | 214·8175 | 14849 |
| | 5 | 1·3900 | 12986 |
| | 27·75 | 216·2075 | 1863 |
| | 5 | ·1114 | 1731 |
| | 27·80 | 216·318\|9 | 132 |
| | 5 | ·111 | 130 |
| | 2\|7·8\|5 | 2\|1\|6\|·4\|30 | 2 |
| | | | 2 |

It is obvious, from an inspection of the work, that if the development had been carried on indefinitely, we should never have obtained a depressed quadratic differing from the following, in the leading portions of the coefficients here exhibited, viz.

$$x^2 + 27{\cdot}8 \dots x + 216{\cdot}4 \dots = 0$$

so that, whatever be the amount of the decimals omitted, we know that of necessity

$$4 \times 216{\cdot}4 \dots > (27{\cdot}8 \dots)^2$$

Hence the remaining roots are imaginary. The only real root is therefore $x = 14{\cdot}95406861$.

6. Required a positive root of the equation

$$4x^3 + 936x^2 + 46674x - 14874 = 0$$

to twelve or thirteen places of decimals.

It is easily seen that the positive root of this equation (for it can have but one) lies between 0 and 1, and that it is nearer to the former limit. The first figure is ·3*

---

* It is worth remarking that when a root is found to lie between 0 and 1, we shall usually be guided to the first figure of it by dividing the abso-

```
4   936              46674              14874·000 (·3166648257063
      1·2              281·16           14086·548
    ————             ————————           ————————
    937·2            46955·16            7874·52000
      1·2              281·52            4724·60764
    ————             ————————            ——————————
    938·4            47236·68            3149·91236000
      1·2                9·3964          2835·66669984
    ————             ——————————          ——————————————
    939·6            47246·0764           314·24566016
        4                9·3968           283·60388489
    —————            ——————————           ————————————
    939·64           47255·4732            30·64177527
        4                5·638464          28·36076065
    ——————           ————————————          ———————————
    939·68           47261·111664           2·28101462
        4                5·638608           1·89071978
    ——————           ————————————           ——————————
    939·72           47266·750272            ·39029484
        24                ·563877            ·37814398
    ———————          ————————————            —————————
    939·744          47267·31414|9            1215086
        24                ·56388               945360
    ———————          ———————————               ——————
    939·768          47267·87803               269726
        24                5638                 236340
    ———————          ——————————                ——————
    939·792          47267·9344|1               33386
         24                564                  33087
    ————————         ——————————                 —————
  9|3|9|·7|9|44      47267·9908                   299
                           37                     283
                     ——————————                   ———
                     47267·994|5                   16
                             4                     14
                     ——————————                    ——
                   47|2|6|7|·9|9|8                  2
                                                   ——
```

The depressed quadratic equation is

$$4x^2 + 939{\cdot}79 \ldots x + 47267{\cdot}99 \ldots = 0$$

and since $4 \times 4 \times 47267{\cdot}99 \ldots < (939{\cdot}79 \ldots)^2$

the remaining roots of the proposed equation are also real. The positive root is $x = {\cdot}3166648257063$.

---

lute number, transposed to the right, by the coefficient of $x$; this coefficient being in fact the trial-divisor suggestive of the root-figure.

7. Required all the roots of the equation

$$4x^3 - 240x^2 + 3996x - 14937 = 0$$

to about eight places of decimals.

This equation has one real positive root at least; and one is plainly situated between 1 and 10. Trying intermediate numbers, we find this root to lie between 5 and 6.

| | | | |
|---|---|---|---|
| 4 | — 240 | 3996 | 14937 (5·246783911 |
| | 20 | — 1100 | 14480 |
| | — 220 | 2896 | 457·000 |
| | 20 | — 1000 | 372·032 |
| | — 200 | 1896 | 84·968000 |
| | 20 | — 35·84 | 72·695296 |
| | — 180 | 1860·16 | 12·272704 |
| | ·8 | — 35·68 | 10·855372 |
| | — 179·2 | 1824·48 | 1·417332 |
| | ·8 | — 7·0976 | 1·265629 |
| | — 178·4 | 1817·3824 | ·151703 |
| | ·8 | — 7·0912 | ·144633 |
| | — 177·6 | 1810·2912 | 7070 |
| | ·16 | — 1·0626 | 5424 |
| | — 177·44 | 1809·228\|6 | 1646 |
| | 16 | 1·063 | 1626 |
| | — 177·28 | 1808·165 | 20 |
| | ·16 | — 124 | 18 |
| | — 177·12 | 1808·04\|1 | 2 |
| | ·24 | — 12 | 2 |
| | —\|1\|7\|7\|·0\|96 | 1807·92 | |
| | | — 1 | |
| | | 1\|8\|0\|7\|·9\|1 | |

The depressed quadratic equation is

$$4x^2 - 177{\cdot}09 \ldots . x + 1807{\cdot}91 \ldots . = 0$$

one root of which evidently lies between 10 and 20, which root we might develope to at least five places of figures; but this is not far enough. The root of the proposed, corresponding to this one, must lie between 15·24 .... and 25·24 ... but whether it be below

20 or above we cannot tell without either finding two figures of the root of the quadratic, or else substituting 10, 20, 30 successively in the proposed equation. We shall take the latter course.

For $x = 10$ the result is +; it is the same for $x = 20$; but for $x = 30$ it is minus. Hence a second root of the proposed lies between 20 and 30. We shall proceed to develope this root.

```
4      — 240         3996          14937 (21·23193870
          80       — 3200          15920
       -----       ------          -----
       — 160          796        — 983 *
          80       — 1600        — 800
       -----       ------          ---
        — 90        — 804        — 183·000
          80            4        — 157·888
          --         ----          ---------
           0        — 800          — 25·112000
           4            8          — 23·588532
          --        -----            ---------
           4        — 792            — 1·523468
           4             2·56          — ·785830
          --        ---------            -------
           8        — 789·44             — ·737638
           4             2·72            — ·707222
          --        ---------              -------
          12        — 786·72                — 30416
            ·8            ·4356             — 23574
          ----      -----------               -----
          12·8      — 786·2844                — 6842
            ·8             4392               — 6286
          ----      -----------                 ---
          13·6      — 785·8452                 — 556
            ·8              148                — 550
          ----      -----------                  ---
          14·4      — 785·830|4                  — 6
            ·12              15
          -----     -----------
          14·52     — 785·815
            ·12            13
          -----     ---------
          14·64     — 7|8|5|·8|0|2
            ·12
          -----
          14·76
              4
          ------
          1|4|·7|64
```

---

* The change of sign here merely shows that a root exists in the interval [0, 20]; which root we have already developed.

The third root of the equation may be obtained by adding the sum of the two now found to $-\frac{240}{4} = -60$, and changing the sign of the result. We shall however merely recur to this property for the purpose of discovering the first figure of the remaining root, which we see is 30; and shall actually develope the root as above: the three results thus obtained will then mutually verify one another.

| | | | |
|---|---|---|---|
| 4 | — 240 | 3996 | 14937 (33·521277383 |
| | 120 | — 3600 | 11880 |
| | — 120 | 396 | 3057 |
| | 120 | 0 | 2376 |
| | 0 | 396 | 681·000 |
| | 120 | 396 | 651·500 |
| | 120 | 792 | 29·500000 |
| | 12 | 432 | 27·724832 |
| | 132 | 1224\| | 1·775168 |
| | 12 | 79·00 | 1·389647 |
| | 144 | 1303·00 | ·385521 |
| | 12 | 80·00 | ·277968 |
| | 156 | 1383·00 | ·107553 |
| | 2·0 | 3·2416 | 97292 |
| | 158·0 | 1386·2416 | 10261 |
| | 2·0 | 3·2432 | 9729 |
| | 160·0 | 1389·4848 | 532 |
| | 2·0 | ·1622 | 417 |
| | 162·0 | 1389·647\|0 | 115 |
| | 8 | 162 | 110 |
| | 162·08 | 1389·809 | 5 |
| | 8 | 32 | 4 |
| | 162·16 | 1389·84\|1 | 1 |
| | 8 | 3 | |
| | 162·24 | 1389·87 | |
| | 4 | 1 | |
| | \|1\|6\|2\|2\|44 | \|1\|3\|8\|9\|·8\|8 | |

Hence the three roots of the proposed equation, to eight places of decimals, are as follow :

$$\begin{array}{lr} & 5\cdot24678391 \\ & 21\cdot23193870 \\ & 33\cdot52127738 \\ \hline \text{Verification} & 59\cdot99999999 = \dfrac{240}{4} \\ \hline \end{array}$$

(114.) Having now sufficiently illustrated the process for developing the roots of a cubic equation, we shall subjoin a few examples as exercises for the student.

8. Required the positive root of the equation

$$x^3 + 60x^2 + 1000x - 1000 = 0$$

to eight places of decimals.

$x = \cdot94551482.$

9. Required the positive root of the equation

$$x^3 + 2x^2 - 5x - 30000 = 0$$

to eight places of figures.

$x = 30\cdot472248.$

10. Required the positive root of the equation

$$x^3 + 10x^2 + 5x - 2600 = 0$$

to ten places of figures.

$x = 11\cdot00679934.$

11. Required the roots of the equation

$$12x^3 - 120x^2 + 326x - 127 = 0$$

to six places of decimals.

$x = \cdot465661.$ Two roots imaginary.

12. Required the roots of the equation

$$x^3 - 7035x^2 + 15262754x - 10000730880 = 0.$$

$x = 1234$, $x = 2345$, $x = 3456$.

13. Required the roots of the equation

$$x^3 - 7x + 7 = 0.$$

$x = 1{\cdot}356895$, $x = 1{\cdot}692021$, $x = -3{\cdot}048916$.

14. Required the roots of the equation

$$x^3 - 21x + 21 = 0$$

to nine places of figures.

$x = 1{\cdot}05608970$, $x = 3{\cdot}96233441$, $x = -5{\cdot}01842411$.

15. Required the roots of the equation

$$5x^3 + 124{\cdot}2x^2 - 338{\cdot}065x - 18606{\cdot}379 = 0$$

to nine or ten places of figures.

$x = 11{\cdot}19733377$. Two roots imaginary.

16. Required the roots of the equation

$$x^3 - 2x - 5 = 0$$

to eighteen places of figures.

$x = 2{\cdot}09455148154232659$. Two roots imaginary.

17. Required the roots of the equation

$$x^3 - 49x^2 + 658x - 1379 = 0$$

to six places of decimals.

$x = 2{\cdot}557351$, $x = 23{\cdot}213112$, $x = 23{\cdot}229637$.

18. Required the roots of the equation

$$4x^3 - 180x^2 + 1896x - 457 = 0$$

to eight places of decimals.

$x = \cdot 24678391,\ x = 16\cdot 23193870,\ x = 28\cdot 52127738.$

19. Required the roots of the equation

$$27x^3 + 27x^2 - 180x + 127 = 0$$

to six places of decimals.

$x = 1\cdot 023562,\ x = 1\cdot 358688,\ x = -3\cdot 382249.$

20. Required the roots of the equation

$$512x^3 + 192x^2 - 10728x + 9409 = 0$$

to eight places of decimals.

$x = \cdot 93108970,\ x = 3\cdot 83733441,\ x = -5\cdot 14342411.$

# CHAPTER III.

## [SUPPLEMENTAL TO THE PRECEDING.]

(115). It has been our endeavour in the foregoing chapter to present, in sufficient detail, the numerical operations for developing the roots of a cubic equation by Horner's method of continuous approximation; and that the simple and uniform character of these operations might be the more distinctly perceived by the student we have disregarded all abbreviating expedients, except the common and obvious ones by which the entrance of superabundant decimals towards the close of the work is provided against. The reader being thus in full possession of the details of the computation we may now proceed with safety to enquire whether by modifying the arrangement of it, in certain circumstances, without destroying the simplicity of its character, we can effect any considerable reduction of the time or space which the performance of it at present requires.

It is clear that in the third column of the work,—that which supplies the absolute numbers of the successive transformed equations—any reduction in the amount of figures is impracticable; so that we may confine our attention to the two preceding columns, or to the formation of the several true and trial-divisors. The derivation of these latter may in all cases be somewhat shortened; but it is only in equations having unity for the coefficient of the highest power that we should recommend, as in quadratic equations under like circumstances, any departure from the general method of procedure already explained. In equations of this kind, and to such may all be reduced by division or otherwise (34), the several trial-divisors may each be readily obtained by putting the square of the last found figure under the preceding true divisor and adding it and the two expressions above it into one sum. Thus a type of the first step is as follows:

$$\begin{array}{rll} & \mathrm{A} & \\ (r+\mathrm{A}_2)\,r = & r^2+\mathrm{A}_2 r & \\ \hline & r^2+\mathrm{A}_2 r+\mathrm{A} & \text{true divisor} \\ & r^2 & \\ \hline & 3r^2+2\mathrm{A}_2 r+\mathrm{A}=\mathrm{A}' & \text{trial-divisor.} \end{array}$$

With this trial-divisor the next root-figure $r'$ is to be found, as usual, and the next step of the work commenced; the value of $\mathrm{A}'_2$ being previously computed from the known relation

$$3r+\mathrm{A}_2=\mathrm{A}'_2 \quad . \quad . \quad . \quad [1]$$

This next step is therefore as follows

$$\begin{array}{rll} & \mathrm{A}' & \\ (r'+\mathrm{A}'_2)r' = & r'^2+\mathrm{A}'_2 r' & \\ \hline & r'^2+\mathrm{A}'_2 r'+\mathrm{A}' & \text{true divisor} \\ & r'^2 & \\ \hline & 3r'^2+2\mathrm{A}'_2 r'+\mathrm{A}'=\mathrm{A}'' & \text{trial-divisor.} \end{array}$$

The trial-divisor $\mathrm{A}''$ determines the next figure $r''$; and another step is completed, as before, after computing $\mathrm{A}''_2$ by the same relation

$$3r'+\mathrm{A}'_2=\mathrm{A}''_2 \text{ or } 3(r+r')+\mathrm{A}_2=\mathrm{A}''_2 \quad . \quad . \quad . \quad [2]$$

And so on to the end. Hence the general type of the first two columns is as follows, the multipliers $r$, $r'$, $r''$, &c. being suppressed, since they are exhibited in the quotient.

$$
\begin{array}{lcl}
 & A & \\
r + A_2 \;.\;.\;.\;. & r^2 + A_2 r & \\
\cline{2-2}
 & r^2 + A_2 r + A & \text{true divisor} \\
 & r^2 & \\
\cline{2-2}
 & A' & \text{trial-divisor} \\
r' + A'_2 \;.\;.\;.\;. & r'^2 + A'_2 r' & \\
\cline{2-2}
 & r'^2 + A'_2 r' + A' & \text{true divisor} \\
 & r'^2 & \\
\cline{2-2}
 & A'' & \text{trial-divisor} \\
r'' + A''_2 \;.\;.\;.\;. & r''^2 + A''_2 r'' & \\
\cline{2-2}
 & r''^2 + A''_2 r'' + A'' & \text{true divisor} \\
 & r''^2 & \\
\cline{2-2}
 & A''' & \text{trial-divisor} \\
 & \&c. \ \&c. &
\end{array}
$$

The quantities $A'_2$, $A''_2$, &c. are all obtained from $A_2$, in combination with the root-figures previously found, as in [1], [2], &c.; or they may be derived one from another by always adding twice the last found root-figure to the quantity last put in the first column: thus

$$(r + A_2) + 2r = A'_2\,; \quad (r' + A'_2) + 2r' = A''_2\,, \quad \&c.$$

And this is in general the most expeditious method of proceeding; and it is so analogous to the mode of operating in the solution of a quadratic equation (104), and in the extraction of the square root, that it is easily kept in the memory. In the process for quadratics, the numbers in the first column are derived, as they become wanted, by adding to the last of those numbers the new figure of the root with *once* the preceding figure prefixed; in the process for cubics the numbers of the first column are derived by adding to the last of those numbers the new figure of the root with *twice* the preceding figure prefixed. Hence the whole operation may be expressed in a rule as follows:

(116.) Rule 1. Put down A, the coefficient of $x$, and a little to the right of it the absolute number, which is to be considered as a dividend, the figures of the root forming the quotient.

2. Place the first figure of the root in the quotient, over which write the coefficient of $x^2$, observing that its units' place be over the units' place of the quotient.
3. Add the root-figure to the quantity thus placed above it: the sum will be the first number of the first column.
4. Multiply this number by the root-figure, add the product to A, and the sum is the first divisor.
5. Under this divisor write the square of the root-figure, add it to the two sums immediately above it, and the result will be the trial-divisor for finding the next figure.
6. Let this new figure be found, and add it, with twice the preceding figure prefixed, to the number already in the first column; and the result will be the next number in that column; which, as before, is to be multiplied by the new figure, and the product added to the trial-divisor for the true divisor; and so on.

NOTE. Instead of placing the coefficient of $x^2$ over the quotient figures, we may arrange the coefficients in a row, as in the general method, and exactly as in that method proceed to determine the first divisor. We shall thus, besides this divisor, get the leading number of the first column, and we may then proceed as in precepts 5 and 6 above.

By substituting these directions for the first four precepts, the rule is shortened and the arrangement of the work rendered more like that of the general method.*

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* The preceding rule is the same in substance as that first published by the author of this treatise in 1823, in his octavo work on *Algebra*, and subsequently in the *Theory of Equations*. As there stated, the rule is as follows:

Put down A the coefficient of $x$, and a little to the right place the absolute number which is to be considered as a dividend, the figures of the root forming the quotient.

Place the first figure of the root in the quotient; above which write the

(117.) 1. As a first example of the application of this rule, let us take that at page 171, viz.

$$x^3+8x^2+6x-75{\cdot}9=0$$

the operation for which, for the purpose of comparison, we shall carry out at length, as at the place referred to.

$$\begin{array}{rll}
 & & 8=A_2 \\
 & 6=A & 75{\cdot}9\ (\ 2{\cdot}4257\ \text{the root} \\
A_2+r=10\ \ldots & 20 & 52 \\
\cline{2-3}
 & 26 & 23{\cdot}9=N' \\
 & 4=r^2 & 22{\cdot}304 \\
\cline{2-3}
 & 50=A' & 1{\cdot}596=N'' \\
A'_2+r'=14{\cdot}4\ \ldots & 5{\cdot}76 & 1{\cdot}239688 \\
\cline{2-3}
 & 55{\cdot}76 & {\cdot}356312=N''' \\
 & {\cdot}16=r'^2 & {\cdot}311827625 \\
\cline{2-3}
 & 61{\cdot}68=A'' & 44484375=N'''' \\
A''_2+r''=15{\cdot}22\ \ldots & {\cdot}3044 & 43716797593 \\
\cline{2-3}
 & 61{\cdot}9844 & 767577407=N''''' \\
 & 4=r''^2 & \&c. \\
\cline{2-2}
 & 62{\cdot}2892=A''' & \\
A'''_2+r'''=15{\cdot}265\ \ldots & 76325 & \\
\cline{2-2}
 & 62{\cdot}365525 & \\
 & 25=r'''^2 & \\
\cline{2-2}
 & 62{\cdot}441875=A'''' & \\
A''''_2+r''''=15{\cdot}2757\ \ldots & 1069299 & \\
\cline{2-2}
 & 62{\cdot}45256799 & \\
 & \&c. &
\end{array}$$

---

coefficient of $x^2$, observing that its units' place be over the units' place of the quotient.

Multiply the value of the quotient figure, taking in those above, by that value; add the product to A, and the sum is the first divisor.

Write the square of the quotient figure just found, under the first divisor, add it to the two sums immediately above, and the result will be the trial-divisor for finding the next figure.

Find now the next figure of the root, and to its value, including those

We shall now add two or three examples worked by the rule, and curtailed of the redundant decimals, as in the general method. The principle of these retrenchments is already too well understood by the student to render any comment upon it necessary here.

2. Find the positive root of the equation

$$x^3 + 5x^2 + 7x = 47$$

Here the first figure of the root is 2, and the operation is as follows:

| | | | |
|---|---|---|---|
| | | | 5 |
| | 7 · | 47 | ( 2·1238 the root |
| 7 . . . . | 14 | 42 | |
| | 21 | 5 | |
| | 4 | 4·011 | |
| | 39 | ·989 | |
| 11·1 . . . . | 1·11 | ·829 | |
| | 40·11 | ·160 | |
| | 1 | ·125 | |
| | 41·23 | 35 | |
| 1\|1\|·3 . . . | ·23 | 33 | |
| | 41·4\|6 | 2 | |
| | 4\|1\|·7 | | |

3. Find the positive root of the equation

$$x^3 + 2x^2 + 3x = 13089030$$

Here the highest denomination of the root is 200.

---

above it, prefix three times the preceding, taking in the value of the figure above it; multiply the result by the last found figure, add the product to the trial-divisor, and we shall have the true divisor. And in the same manner are the succeeding divisors to be obtained.

This last precept is more complicated in its expression than that marked 6 above, which supplies its place. The operation of trebling the root, &c. at every step is better exchanged for adding its double to the preceding number. See example 7.

| | | |
|---|---|---|
| | | 2 |
| | 3 | 13089030 (235 the root |
| 202 . . . . | 40400 | 80806 |
| | 40403 | 500843 |
| | 4 | 419289 |
| | 120803 | 815540 |
| 632 . . . . | 1896 | 815540 |
| | 139763 | |
| | 9 | |
| | 159623 | |
| 697 . . . . | 3485 | |
| | 163108 | |

By carrying the first two columns one step further, we have, for the depressed equation containing the other roots diminished by 235, the quadratic

$$x^2+707x+16618=0$$

the roots of which are real.

4. Find the positive root of the equation

$$x^3+24x=68$$

Here the first figure of the root is found to be 2

| | | |
|---|---|---|
| | 24 | 68 (2·3158 the root |
| | 4 | 56 |
| | 28 | 12 |
| | 4 | 11·367 |
| | 36 | ·633 |
| 6·3 . . . . | 1·89 | ·399 |
| | 37·89 | ·234 |
| | 9 | ·200 |
| | 39·87 | 34 |
| 6\|·9 . . . . | 7 | 32 |
| | 39·9\|4 | 2 |
| | 4\|0\|0 | |

5. Find the positive root of the equation

$$x^3 + x^2 = 500$$

to about eight places of decimals.

In this example the first figure of the root is 7

```
                              1
                   0          500 (7·61727975 the root
8 . . . .         56          392
                  ——          ———
                  56          108
                  49          104·736
                  ——          ———————
                 161            3·264
22·6 . . . .     13·56          1·887181
                 ——————         ————————
                 174·56         1·376819
                     36         1·323862
                 ——————         ————————
                 188·48           52957
23·81 . . . .      2381           37859
                 ————————         —————
                 188·7181         15098
                        1         13251
                 ————————         —————
                 188·9563          1847
2|3·8|4 . . . .      1669          1704
                 ——————————        ————
                 189·123|2          143
                 —————————          133
                 189·290            ———
                       5             10
               ——————————             9
               1|8|9|·2|9|5          ——
                                      1
```

6. Required a positive root of the equation

$$x^3 - 17x^2 + 54x = 350$$

to about 10 places of figures.

Here the first denomination of the root is 10.

| | | |
|---|---|---|
| | | — 17 |
| | 54 | 350 (14·95406861 the root |
| — 7 . . . . | — 70 | — 160 |
| | — 16 | 510 |
| | 100 | 328 |
| | 14 | 182 |
| 17 . . . . | 68 | 170·379 |
| | 82 | 11·621 |
| | 16 | 10·740875 |
| | 166 | ·880125 |
| 25·9 . . . . | 23·31 | ·865276 |
| | 189·31 | 14849 |
| | ·81 | 12986 |
| | 213·43 | 1863 |
| 27·75 . . . . | 1·3875 | 1731 |
| | 214·8175 | 132 |
| | 25 | 130 |
| | 216·2075 | 2 |
| 2\|7·8\|5 . . . . | ·1114 | 2 |
| | 216·318\|9 | |
| | 2\|1\|6\|·4\|30 | |

7. Required a positive root of the equation

$$x^3 + 24{\cdot}84x^2 - 67{\cdot}613x = 3721{\cdot}3758$$

to about 10 places of figures.

Here the highest denomination of the root is 10.

```
                                          24·84
                 —67·613            3761·2758 (11·19722208 the root
34·84 * . . . . 348·4               2807·87
 21             ________            _________
                 280·787             953·405
                 100                  785·027
                ________            _________
                 729·187              168·3788
55·84 . . . .     55·84                84·7661
 2·1            ________             ________
                 785·027               83·6127
                    1                  77·283513
                ________             __________
                 841·867                6·329187
57·94 . . . .      5·794                6·050544
   29           ________             __________
                 847·661                 ·278643
                       1                 ·259437
                ________             __________
                 853·465                  19206
58·23 . . . .      5·2407                 17296
   187          _________                 _____
                 858·7057                  1910
                       81                  1730
                _________                  ____
                 863·9545                   180
5|8·4|2 . . . .     ·4089                   173
                _________                   ___
                 864·363|4                    7
                _________                     6
                 864·772                      _
                      17                      1
                _________
                 864·78|9
                ________
                |8|6|4|·8|1
```

---

* In this example the double of each root-figure, employed in deriving the numbers of this column one from another, are actually exhibited, with the new figure annexed to each. But the formal insertion of them may evidently be dispensed with.

The following examples are added for exercise in the preceding rule:

| | *Equations.* | *Positive roots.* |
|---|---|---|
| 8. | $x^3 - 6x^2 + 18x = 22$ | $x = 2{\cdot}3274$ |
| 9. | $x^3 + 4x^2 + 2x = 2328$ | $x = 12$ |
| 10. | $x^3 + 8x = 34648584$ | $x = 236$ |
| 11. | $x^3 + 2{\cdot}5x^2 + 2x = {\cdot}5$ | $x = {\cdot}1974$ |
| 12. | $x^3 + 4{\cdot}73x = 1{\cdot}746$ | $x = {\cdot}3594$ |
| 13. | $x^3 + 9x^2 + 4x = 80$ | $x = 2{\cdot}4721$ |
| 14. | $x^3 - 12x + 8 = 0$ | $x = {\cdot}6945928$ |
| 15. | $x^3 + x^2 + x = 100$ | $x = 4{\cdot}26442997$ |

### ON THE EXTRACTION OF THE CUBE ROOT.

(118.) From the preceding method of extracting the roots of cubic equations may be derived a new method of extracting the cube root of numbers, which will be much more easy and concise than the method usually given.

1. Suppose, for example, it were required to extract the cube root of the number 12326391, or, which is the same thing, to find the real root of the cubic equation

$$x^3 + 0x^2 + 0x = 12326391.$$

By proceeding according to the method in art. (116) the process will be as follows:

```
                                  0
                 0         12326391 (231 the root
  2 . . . .      4         8
                 —         ————
                 4         4326
                 4         4167
                 —         ———————
                12          159391
 63 . . . .     189         159391
                ———        ———————
               1389        ———————
                  9
                ———
               1587
691 . . . .      691
               ——————
               159391
```

From inspecting the above operation, it will be obvious that some of the work is superfluous; thus, the first trial-divisor, 12, might have been easily found at once, by multiplying the square of the root-figure 2, by 3; also, since the numbers that are placed under the trial-divisors to be added thereto, always have two figures to the right, when the addition is performed they are written down again; but this repetition would be avoided if these two numbers were placed *at first* a line lower down, and only the other figures placed *immediately under* the trial-divisor; but then, in afterwards adding the square of the new figure, these two figures must be repeated *twice* in the addition, so that we have the following

### *New Method to extract the Cube Root of any given Number.*

(119.) Divide the given number into periods of three figures each, as in the common method, and find the nearest cube to the first period; subtract it therefrom, and put the root in the quotient: then thrice the square of this root will be the trial-divisor for finding the next figure.

Draw a line a little below the trial-divisor; multiply the new figure, with thrice the preceding prefixed, by the new figure, and place the first two figures of the product *below* this line, to the right of the trial-divisor, and the others *above* the line; add them to the trial-divisor, and the sum will be the true divisor.

Under this divisor write the square of the last root figure, which add to the two sums above, repeating the two final figures of the divisor, and the result is the next trial-divisor; the true divisor is found as before, &c. *

NOTE. After the first or second decimal place in the root is found, the square of the root-figure used in forming the trial-divisor may be omitted, and also those two figures that would fall below the line in forming the true divisor; as the value of these figures will be too small for their omission to affect the truth of the result. But, if the number of decimals in the root is required to be very great, these omissions must not be made till after the third or fourth decimal in the root is found.*

---

* The numbers in the first column of the work may be obtained one from another without trebling the root after the first step. It will be sufficient to add twice the preceding root-figure to the preceding number, and then to annex the new root-figure to the result. This corresponds to the plan adopted in the process for the square root; where the numbers of the first column are derived, one from another, by adding *once* the preceding root-figure to the preceding number, and then annexing the new figure to the result (104.) See example 2 following.

The preceding example, by this rule, will stand thus:

```
               12              12326391 (231
                1              8
63 . . . .   ----              ----
             1389              4326
                9              4167
             ----              ------
             1587               159391
                6               159391
691 . . . .  ------            -------
             159391            -------
```

2. Extract the cube root of the number 673373097125.

Here the nearest root of the first period, 673, is 8; hence the operation is as follows:

```
                   192            673373097125 (8765
                   17             512
247† . . . .      -----           ------
14                20929           161373
                     49           146503
                  -----           --------
                  22707            14870097
                    156            13718376
2616 . . . .      -------         ----------
  12              2286396           1151721125
                       36           1151721125
                  -------         ------------
                  2302128         ------------
                     1314
26285 . . . .     ---------
                  230344225
                  ---------
```

---

* At whatever divisor these contractions take place, as many more decimals of the root will be obtained as there are figures in this divisor, *minus* one, although the last decimal thus obtained, if the root has been extended to fourteen or fifteen places, is not always to be depended upon.

† We have actually inserted the doubles of the successive root-figures

3. **Extract the cube root of 3 to three places of decimals.**

```
              3          3 (1·442
              1          1
3·4 . . . .  ———        ——
             4·36        2

              ·16       1·744
             ————       —————
             5·88        ·256
              ·17        ·242
4·2 . . . .  ———        ————
             60|5          14
              —            12
             6|2           —
                            2
```

4. **Extract the cube root of 3 to six places of decimals.**

```
              3          3 (1·442249
              1          1
3·4 . . . .  ———        ——
             4·36        2
              ·16        1·744
             ————       ——————
             5·88         ·256
              ·16         ·241984
4·24 . . .   —————      ————————
             6·0496        14016
                 16        12459
             ——————       ——————
             6·2208         1557
                 86         1248
|4·3|2 . . . ——————        —————
             6·229|4         309
             ———————         250
             6·238           ———
                 1            59
             ——————           56
             6·23|9           —
             ——————            3
             6|2|4
```

---

in this column to show more distinctly the derivation of the numbers after the first, according to the directions in the foot-note at page 202. But the formal insertion of them may obviously be dispensed with.

5. Extract the cube root of 9 to about fourteen or fifteen places of decimals.

```
                12                 9 (2·080083823051904*
                                   8
6·08 ......  ________              _
             12·4864               1
                  64                ·998912
             _______                _______
             12·9792                  1088
                   4                  1038375936512
6·24008 .... ____________             _____________
             12·9796992064              49624063488
                        64              38940651420
             _____________              ___________
             12·9801984192              10683412068
                    187207              10384192682
6|·24|02|4... ______________            ___________
             12·980217139|9               299219386
             ______________               259604919
             12·980235861                 _________
                     4992                  39614467
             ____________                  38940738
             12·98024085|3                 ________
             _____________                   673729
             12·98024585                     649012
                      12                     ______
             ___________                      24717
             12·9802459|7                     12980
             ____________                     _____
            12|9|8|0|2|4|6|1                  11737
                                              11682
                                              _____
                                                 55
                                                 52
                                                 __
                                                  3
```

---

* Another figure might have been obtained, viz. 2; but, on account of the extent to which the root has been carried, this figure could not have been depended on as true; all the fifteen places, however, that we have found are true to the last figure.

6. Extract the cube root of 6 to about six places of decimals. Ans. 1·817120.

7. Extract the cube root of 18609625. Ans. 265.

8. Extract the cube root of 5 to six places of decimals. Ans. 1·709975.

9. Extract the cube root of 469640998917. Ans. 7773.

10. Extract the cube root of 2. Ans. 1·25992105.

11. Extract the cube root of 1·25992105. Ans. 1·08005974.

12. Extract the cube root of 6692234354139671875. Ans. 1884475.

### *Scholium.*

In the preceding chapters on the solution of Cubic Equations we have endeavoured to present the subject with details sufficiently ample for all the demands of actual practice. And it has been shown how all the roots of such equations may be developed without the necessity of submitting them to a previous analysis; the circumstance that relieves us from this trouble being the necessary existence of one real root in every cubic, of which the situation is always discoverable by successive substitution (112.) But it would be a desirable improvement if, after having developed this root, we could make it contribute to the determination of the other roots, and thus spare us the labour of the additional developments. We have investigated a formula for this purpose, which exhibits two roots of a cubic in terms of the third root; and by help of which the remaining roots may be expeditiously calculated when one is found; or the forms of these roots exhibited when they are imaginary. The student is referred, for this formula, to Note (A) at the end.

# CHAPTER IV.

## ON THE SOLUTION OF BIQUADRATIC EQUATIONS.

(120.) The method of developing a root of a biquadratic equation, or indeed of any numerical equation whatever, cannot but suggest itself to the student after what has been taught, in the preceding chapters of the present section, respecting equations of the second and third degrees. For the principles from which the rules established in those chapters have been derived are evidently quite unrestricted in their application. These principles may be briefly described as consisting simply of a series of successive transformations performed upon the proposed equation by diminishing its roots by the figures, one after another, of that particular root we wish to develope: each figure, after the first, being suggested by the coefficient of $x$ in the preceding transformation. This coefficient has been called the trial-divisor, and tends, as the work proceeds, and from causes already explained (105), to approach nearer and nearer to the true divisor of which it is a part. In other words, the correction of this trial-divisor becomes gradually more and more insignificant, till at length it may be altogether neglected, as in the examples given in the preceding chapters.

If $r$ be the first figure of a root of the biquadratic equation

$$A_4x^4 + A_3x^3 + A_2x^2 + Ax = N$$

then, diminishing by $r$, the transformed equation in $x - r$ will be

$$A_4x'^4 + A'_3x'^3 + A'_2x'^2 + A'x' = N'$$

The first figure $r'$ of a root of this, or the second figure of the root to be developed, is suggested by dividing N′ by the trial-divisor A′; and its accuracy is afterwards tested by completing with it the true divisor

$$A_4r'^3 + A'_3r'^2 + A'_2r' + A'$$

as in cubic equations. The second figure being thus determined, the preceding transformed equation is diminished by it, and a second transformed equation derived, containing the roots of the original diminished by $r + r'$. The coefficient A″ of $x''$ in this equation suggests the third figure; which being verified, the roots are diminished by it, and a third transformed equation derived; and so on, agreeably to the details of the investigation given at (105), and which, as remarked above, is seen at once to be unaltered in its character, whatever be the degree of the equation submitted to it.

All the operations implied in these successive steps, when conducted as already explained, are remarkable for their blending together in one continuous process. When the first figure is found, the trial-divisor for the next spontaneously presents itself in the course of the work for deducing the transformed equation; the true divisor, formed by aid of the figure suggested by this trial-divisor, in like manner presents itself in our route towards the next transformation; and so on continuously. The entire series of operations may be described in words as follows:

(121.) Rule 1. Arrange the coefficients of the given equation in a row.

2. Multiply the first root-figure previously found by the first coefficient, and add the product to the second coefficient: multiply the result by the same root-figure, and add the product to the third coefficient: multiply this new result by the same root-figure, and add the product to the fourth coefficient. The sum will be the true divisor.

3. Return now to the first coefficient, and perform a similar

course of operations with the same root-figure, stopping as before at the result under the fourth coefficient. This result will be the trial-divisor A′.

4. Repeat again this process, stopping here, however, at the result under the *third* coefficient.
5. Commence anew, as above; stopping, however, at the first result—that under the second coefficient. And the oblique row of results thus obtained, when preceded by the first of the original coefficients, will be the coefficients of the first transformed equation.
6. By means of the fourth of these coefficients—that is, of the trial-divisor A′, find the next figure of the root; and proceed with this new figure, and with the new coefficients, through exactly the same courses of operations as with the first figure and the original coefficients. We shall, as before, arrive at the true divisor at the end of the first of these courses; and at a new trial-divisor at the end of the second; and these, with the two results still remaining, furnish as before the successive coefficients, in reverse order, of the new transformed equation; with which, after the third figure of the root is found from the fourth of these coefficients, A″, we are to proceed as at first: and so on till the root is completed, or till any proposed degree of approximation is reached.

(122.) The above are the precepts which have been followed, virtually, in the solution of equations of the second and third degrees; and which are efficient be the degree of the equation any whatever. The successive transformations, through which we pass as the successive figures of the root become determined, are derived one from another by courses of operations similar to those described above. In an equation of the second degree there are *two* such courses; in an equation of the third degree, *three;* in an equation of the fourth degree, *four;* and general in an equation of the *n*th degree, *n*. And in all cases the first course terminates in the true divisor corresponding to the figure under trial, while the second course terminates in the trial-divisor suggestive of the next figure.

Thus one general rule, and one uniform series of recurring operations, serve for the solution of all classes of numerical equations when the places of the roots, or the figures with which they commence, are ascertained by a previous analysis.

It might seem, therefore, that the investigation of this one rule would suffice for all the demands of a complete treatise on the practical solution of equations of all degrees; and consequently that the more circuitous course adopted in the present section, of discussing separately, and successively, the inferior classes of equations from the second up to the fourth, might be advantageously superseded by a shorter and more comprehensive method of procedure.

But, independently of the advantage to the student of unfolding gradually, and of first illustrating in their more simple applications, the general rules and principles of science, there are other and still stronger reasons to justify the plan upon which we have here proceeded, and the limits which we have set to it; reserving the consideration of the problem of continuous approximation in all its generality for separate discussion in a distinct treatise on Equations of the Higher Orders. For at the point where we have now arrived, it is *possible* for perplexities to arise in the course of our developments which, for want of certain theories not entered upon in the present work, might greatly retard the progress of evolution. The exposition of these theories is better adapted for a work of somewhat higher pretensions than the present Introduction; and it will accordingly be found in the volume referred to, where the principles here established are more fully and completely carried out.

We shall now proceed to some applications of the foregoing rule; and shall more distinctly explain the nature of the perplexities just adverted to when we come to example 4.

In working these examples we shall employ the contractions, already made use of in developing the roots of quadratic and cubic equations for the purpose of precluding the entrance of useless decimals. These contractions are regulated by one general principle: they commence with the final column or that which furnishes the successive absolute numbers. This proceeds uncontrolled till we reach a result containing at least one place of figures more than there are places still to be determined in the root;[*] the column is not allowed to make any advance to the right after this result is arrived at. To provide against such an advance the results in the preceding column must not only not advance but must *recede* a place at every step: and the column next in order, to accommodate itself to this recession, must recede in like manner, and so on.

We have further to observe that in order the more clearly to trace the oblique rows of figures belonging to each transformation, we shall place against each of the lines immediately over the first row, the number **1**; against each of the lines over the next row the number **2**; and so on: a practice we would recommend to the student in all long operations of this kind, as it will distinctly mark the several transformations and will facilitate the revision of any suspected step.[†]

## EXAMPLES.

(123.) 1. One root of the biquadratic equation

$$x^4 + 3x^3 + 2x^2 + 6x - 148 \cdot 6 = 0$$

is found by Sturm's theorem to lie between 2 and 3; required the development of that root to six places of decimals.

---

[*] In certain peculiar cases, however, we shall not obtain so many additional figures as this statement would imply; but only a number of figures far short of that in the absolute number at which our contractions commence. An instance of this will be seen in the developments of example 4 at page 227.

[†] A contrivance analogous to this is adopted by Mr. De Morgan, in his article on *Involution*, in the 'Penny Cyclopædia.'

```
1     3        2          6          148·6   (2·734400
      2       10         24           60
      —        —          —          ———————1
      5       12         30           88·6000
      2       14         52           22·9731
      —        —         —1          ———————2
      7       26         82            5·6269
      2       18         36·533        4·8977
      —       —1        ————          ——————3
      9       44        118·533        ·7292
      2        8·19      42·609        ·6627
     —1       ————      ——————2       —————4
     11       52·19     161·142         665
       ·7      8·68       2·114         664
    ————      ————      ——————        ————
     11·7     60·87     163·25|6           1
       ·7      9·17       2·13
    ————      ————2     ——————3
     12·4     70·04     165·39
       ·7       41         ·28
    ————      ————      ——————
     13·1     70·4|5    165·6|7
       7        4          3
    ————2     ————      ——————4
     13·8     70|·9     1|6|6|·0
        3     ————3
    ————      |7|1
   1|3|·83
```

2. One root of the biquadratic equation

$$x^4 - 3x^2 + 75x - 10000 = 0$$

is found to lie between 9 and 10; required the development of it to eight places of figures.

```
1     0        —3        75           10000 (9·8860027
      9        81        702          6993
      —        —         ———          ————————
      9        78        777          3007·0000
      9        162       2160         2677·5616
      —        ——        ——— 1        ————————— 2
      18       240       2937         329·4384
      9        243       409·952      306·1662
      —        —— 1      ———————      ——————— 3
      27       483       3346·952     23·2722
      9        29·44     434·016      23·2616
      —— 1     ————      ——————— 2    —————— 4
      36       512·44    3780·968       106
      ·8       30·08     46·110          78
      ———      ————      ———————        ——
      36·8     542·52    3827·07|8       28
      8        30·72     46·36           27
      ———      ————— 2   —————— 3        —
      37·6     573·24    3873·44          1
      8        3·14      3·50
      ———      ————      ———————
      38·4     576·3|8   3876·9|4
      8        3·1       3·5
      ——— 2    ————      —————— 4
      39·2     579|·5    3|8|80·4
      8        3
      ———      ————— 3
      3|9|3    5|8|3
```

3. Required the analysis and solution of the equation

$$x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$$

*Analysis of the Equation.*

$$X = x^4 - 80x^3 + 1998x^2 - 14937x + 5000$$

$$X_1 = 4x^3 - 240x^2 + 3996x - 14937$$

$$\begin{array}{r|r}
\overset{1}{4} & -\ 240 + \ 3996 - 14937 \\
-\overset{-20}{80} & -3996 + 44811 - 20000 \\
\hline
 & 4800 - 79920 + 298740 \\
 & -3996 + 44811 - \ 20000 \\
\hline
 & 804 - 35109 + 278740
\end{array}$$

$$\therefore\ X_2 = 804x^2 - 35109x + 278740$$

The roots of the equation $X_2 = 0$ are both positive (93); and the sum of them, $\frac{35109}{804}$, is a little above 40. Their situations are easily found to lie between 10 and 20, and between 30 and 40.

Now by the rule of Descartes the proposed equation has no negative root; and by (47) 81 is a superior limit to the positive roots. Hence, substituting 0, 1, 10, 20, &c. successively for $x$ in the three polynomials X, $X_1$, $X_2$, we have the following results:

| | $X$ | $X_1$ | $X_2$ | |
|---|---|---|---|---|
| $x=0$ . . . | + | − | + | two variations |
| 1 . . . | − | − | + | one variation |
| 10 . . . | − | + | + | one variation |

Hence the equation has one root and only one between the limits [0, 10], since within these limits the sign of $X_2$ is invariable. Continuing the substitutions we have

| | | | |
|---|---|---|---|
| $x=20$ . . . | + | + | − |
| 30 . . . | + | − | − |
| 40 . . . | + | + | + |
| $\infty$ . . . | + | + | + |

In the interval [10, 20] the change of sign in X shows that *one* root must exist: there cannot be *three*, because the preceding and subsequent changes of sign in $X_1$ show that the sign of $X_1$ is invariable throughout the interval [10, 20] (73).

As the sign of $X_2$ is invariable throughout the interval [20, 30], and as no variation is lost in the signs up to $X_2$, there exists no root in this interval. In like manner, as no variation is lost in the interval [40, ∞], there exists no root so great as 40. We have therefore only to examine the interval [30, 40] in order to ascertain whether two roots of the equation occur in this interval, or whether the two remaining roots are imaginary.

For this purpose let us develope, figure by figure, alternately, the roots of $X_2 = 0$ and $X_1 = 0$, which lie in the interval under examination; with a view to ascertain which of these polynomials arrives first at zero, whilst $x$ passes from 30 to 40.

| | *Development of x in* $X_2=0$ | | | *Development of x in* $X_1=0$ | | |
|---|---|---|---|---|---|---|
| 804 | —35109 | —278740 (33·2 | 4 | —240 | 3996 | 14937 (33·5 |
| | 24120 | —329670 | | 120 | —3600 | 11880 |
| | — | — 1 | | — | — | — 1 |
| | —10989 | 50930 | | —120 | 396 | 3057 |
| | — 1 | 46629 | | 120 | 0 | 2376 |
| | 13131 | — 2 | | — | — 1 | — 2 |
| | 2412 | 4301·00 | | 0 | 396 | 681·000 |
| | — | 3607·16 | | 120 | 396 | 651·500 |
| | 15543 | — 3 | | — 1 | — | — 3 |
| | — 2 | 693·84 | | 120 | 792 | 29·500 |
| | 17955 | | | 12 | 432 | |
| | 80·8 | | | — | — 2 | |
| | — | | | 132 | 1224 | |
| | 18035·8 | | | 12 | 7900 | |
| | | | | — | — | |
| | | | | 144 | 130300 | |
| | | | | 12 | | |
| | | | | — 2 | | |
| | | | | 156 | | |
| | | | | 2·0 | | |
| | | | | — | | |
| | | | | 158·0 | | |

Consequently $X_2$ vanishes before $X_1$, so that in passing continuously from $x = 30$ to $x = 40$, $X_1$ continues *minus* after $X_2$ has become plus. Therefore immediately before $X_2$ vanishes the signs are

$$\begin{array}{lccc} & X_1 & X_2 & X_3 \\ & - & - & \\ & & & + \\ \text{and immediately after} & - & + & \end{array}$$

the sign of $X_3$ being inferred from the sign which $X_1$ takes when $X_2$ vanishes (67).

It appears, therefore, that the evanescence of $X_2$ causes the loss of a variation in the series $X_2$, $X_3$, $X_4$. This is the only loss (or gain) that can take place in this series within the interval $x = 30$; $x = 40$, since $X_2$ vanishes but once in this interval (73). But another variation is lost in the same interval in the former part of the series, that is, in $X$, $X_1$, $X_2$. Hence *two* variations are lost in passing from $x = 30$ to $x = 40$; and thus the equation has two roots in this interval. The roots are therefore all real, and situated in the following intervals, viz.

$$[0, 1]; \quad [10, 20]; \quad [30, 40]; \quad [30, 40]$$

*Development of the First Root.*

```
1    — 80          1998           — 14937             — 5000 (·3509870
        ·3         — 23·91            592·227          — 4303·4319
     ——————        ————————       ——————————          ——————————— 1
     — 79·7        1974·09        — 14344·773           — 696·5681
        ·3         — 23·82            585·081           — 683·1781
     ——————        ————————       ——————————— 1         ————————— 2
     — 79·4        1950·27        — 13759·692            — 13·3900
        ·3         — 23·73             96·130            — 12·2093
     ——————        ———————— 1     ——————————             ——————— 3
     — 79·1        1926·54        — 13663·56|2            — 1·1807
        ·3          — 3·94             95·94                1·0850
     —————— 1      —————————      ——————————— 2            ———————
     — 78·8        1922·6|0       — 13567·62                 — 957
         5          — 3·9               1·72                 — 949
  ————————————     —————————      ——————————                 —————
  |— 7|8|·8        1918|·7        — 13565|·90                  — 8
                    — 4                2
                ——————————— 2     ——————— 3
                |1|·9|15          — 13564
                                       1
                                  ———————————
                                — 13|5|6|3
```

Hence the first root is

$$x = \cdot 3509870$$

### *Development of the Second Root.*

```
1     —80        1998       —14937       —5000 (12·7564418
       10        —700        12980      —19570
       —         ——          ————        ———— 1*
      —70        1298        —1957       14570
       10        —600         6980       10534
       —         ——          ——— 1       ————— 2
      —60         698         5023       4036·0000
       10        —500          244       3737·3441
       —         —— 1        ———         ————— 3
      —50         198         5267        298·6559
       10         —76         —100        264·5973
       — 1        ——         ——— 2       ———— 4
      —40         122         5367         34·0586
        2         —72        —27·937       31·7230
       —          —          ————        ———— 5
      —38          50        5339·063       2·3356
        2         —68        —42·931        2·1147
       —          — 2        ————          ———
      —36         —18        5296·132        2209
        2        —21·91       —4·186         2114
       —          ——         ————            ——
      —34        —39·91      5291·94|6         95
        2        —21·42       —4·26            53
       — 2        ——         ——— 4             —
      —32        —61·33      5287·69           42
       ·7        —20·93        —52             42
      ——          —— 3       ————              —
     —31·3       —82·26      5287·1|7
        7         —1·46        —5
      ——          ——         ——— 5
     —30·6       —83·7|2     5|2|8|6|7
        7         —1·5
      ——          ——
     —29·9       —85|·2
        7          —2
      —— 3       ——— 4
     —29·2       |—8|7
        5
      ———
   |—2|9|·2
```

Therefore the second root is

$$x = 12·7564418$$

---

* The change of sign in the absolute number of the transformation **1** implies that, in passing from $x = 0$ to $x = 10$, a root of the proposed equation is passed over; that is, that a root exists in the interval [0, 10]. This is the root already developed.

### *Development of the Third Root.*

```
1  — 80       1998        — 14937       — 5000 (32·0602909
     30     — 1500          14940           90
     ——       ————          —————         ———— 1
   — 50        498              3       — 5090
     30      — 600         — 3060       — 4986
     ——       ————          ————— 1       ———————— 2
   — 20      — 102         — 3057        — 1040000
     30        300            564        — 1035064
     ——       ———— 1        —————         ———————— 3
     10        198         — 2493           — 4936
     30         84            740           — 3394
     —— 1     ————          ————— 2          —————
     40        282         — 1753           — 1542
      2         88          27·893          — 1526
     ——       ————         ———————           —————
     42        370      — 1725·10|7           — 16
      2         92          28·07             — 14
     ——       ———— 2       ——————— 3            ——
     44        462        1697·04              — 2
      2       2·88              9
     ——       ————      ——————————
     46     464·8|8    1|6|9|6|·9|5
      2        2·9
     —— 2     ————
     48      467|·8
     ·06         2
   ——————     ——— 3
   4|8|·06    4|70
```

Therefore the third root is

$$x = 32·0602909$$

The remaining root may be found by adding the sum of the roots now determined to — 80, the coefficient of the second term of the proposed equation, and changing the sign of the result. But we shall seek this root by direct development, as above.

The first figure of this remaining root is 3, in the tens' place, as we have previously discovered. Moreover this root is greater than the root just determined; since, in the course of developing it, we have not passed over any other root above 30, thus rendering that root *negative;* for the successive transformed equations **2**, **3**, &c. exhibit the very same signs as the first transformed equation **1**, which involves the roots of the proposed, each diminished by 30. If the above had been the development of the greater root in the interval [30, 40], then, after the transformation **1**, there must have occurred, sooner or later, a change of sign in the final column, marking the place at which the two roots separate. The *original* equation loses two variations of sign by the first transformation; and this shows that *two* roots are changed from positive to negative by being diminished by 30; because the roots, as well in the transformed as in the original equation, being all real, the rule of Descartes makes known the character of the roots at every step. But, without regarding the signs of the several transformed coefficients, we may infer immediately from the analysis, or rather from the development of $x$ in $X_1 = 0$, at page 215, that since the root thus developed lies between the roots of $X = 0$ in the interval [30, 40], by art. (56), — we say, we may infer from this that the remaining root must exceed 33·5 . . .

As then this remaining root must lie between 33 and 40, the first figure of it, when diminished by 30, must lie between 3 and 10; that is to say, such must be the situation of a root of the transformed equation **1**. We shall therefore commence with this transformed equation; and trying the substitutions 3, 4, . . . we find 4 to be the first figure of the root sought. Hence we proceed as follows:

### *Development of the Fourth Root.*

| | | | | |
|---|---|---|---|---|
| 1 | 40 | 198 | — 3057 | — 5090 (4·8322803 |
| | 4 | 176 | 1496 | — 6244 |
| | —— | —— | —— | —— 2 * |
| | 44 | 374 | — 1561 | 1154·0000 |
| | 4 | 192 | 2264 | 1086·8416 |
| | —— | —— | —— 2 | —— 3 |
| | 48 | 566 | 703 | 67·1584 |
| | 4 | 208 | 655·552 | 62·3517 |
| | —— | —— 2 | —— | —— 4 |
| | 52 | 774 | 1358·552 | 4·8067 |
| | 4 | 45·44 | 692·416 | 4·2154 |
| | —— 2 | —— | —— 3 | —— 5 |
| | 56 | 819·44 | 2050·968 | 5913 |
| | ·8 | 46·08 | 27·421 | 4219 |
| | —— | —— | —— | —— |
| | 56·8 | 865·52 | 2078·38\|9 | 1694 |
| | 8 | 46·72 | 27·47 | 1687 |
| | —— | —— 3 | —— 4 | —— |
| | 57·6 | 912·24 | 2105·86 | 7 |
| | 8 | 1·78 | 1·84 | 6 |
| | —— | —— | —— | —— |
| | 58·4 | 914·0\|2 | 2107·7\|0 | 1 |
| | 8 | 1·8 | 1·8 | |
| | —— 3 | —— | —— 5 | |
| | 59·2 | 915·\|8 | 2109·5 | |
| | 3 | 2 | 2 | |
| | —— | —— 4 | —— | |
| | 5\|9\|·2 | \|9\|1\|8 | 2\|1\|0\|9\|7 | |

---

* The change of sign in the absolute number of this transformed equation shows us that a root of **2** lies between 0 and 4; or, which amounts to the same thing, that a root of the original equation lies between 30 and 34. This is the root 32·06 . . . previously developed.

Consequently the fourth root is

$$x = 34{\cdot}8322803$$

Adding the others, $32{\cdot}0602909$

$12{\cdot}7564418$

${\cdot}3509870$

there results for the sum $80{\cdot}0000000$

which result verifies each of the roots

4. Required the analysis and solution of the equation

$$x^4 + 312x^3 + 23337x^2 - 14874x + 2360 = 0$$

*Analysis of the Equation.*

$$\mathrm{X} = x^4 + 312x^3 + 23337x^2 - 14874x + 2360$$

$$\mathrm{X}_1 = 4x^3 + 936x^2 + 46674x - 14874$$

$$\begin{array}{r|rrrrrr}
\overset{1}{4} & +\,936 & + & 46674 & - & 14874 \\
\overset{78}{312} & -\,46674 & + & 44622 & - & 9440 \\
\cline{2-6}
 & 73008 & + & 3640572 & - & 1160172 \\
 & -\,46674 & + & 44622 & - & 9440 \\
\cline{2-6}
 & 26334 & + & 3685194 & - & 1169612
\end{array}$$

$$\therefore \mathrm{X}_2 = 26334x^2 + 3685194x - 1169612$$

By (47) 2 is a superior limit to the positive roots of the proposed equation. Hence, substituting in succession 0, 1, 2 for $x$ in the three polynomials $X$, $X_1$, $X_2$, we have the following results, namely,

| | $X$ | $X_1$ | $X_2$ |
|---|---|---|---|
| $x = 0$ .... | $+$ | $-$ | $-$ |
| 1 .... | $+$ | $+$ | $+$ |
| 2 .... | $+$ | $+$ | $+$ |

These results inform us that the equations $X_1 = 0$, $X_2 = 0$ have each a positive root between 0 and 1; and, by the rule of Descartes, no other positive root can exist in either of these equations. Hence 1 is beyond the limit of the positive root of $X_2 = 0$, and as no variation is lost in passing from 1 to 2 in the foregoing table of signs we conclude that the proposed equation has no roots between 1 and 2. Hence, if real positive roots exist, they must both lie between 0 and 1. This, therefore, is the interval we have to examine; and in order to determine its character we must, as in the former example, ascertain which of the two polynomials, $X_1$ or $X_2$, vanishes first, in consequence of $x$ proceeding continuously from $x = 0$, to $x = 1$.

The root of the equation $X_1 = 0$ we have already developed at page 184 to thirteen places of decimals, and have found it to be

$$x = \cdot 3166648257063$$

Let us now develope the root of $X_2 = 0$ to the same extent:

| | | |
|---|---|---|
| \|26\|33\|4 | 3685194 | 1169612·00 (·31666482570627 |
| | 7900·2 | 110792826 |
| | 3693094·2 | 616837400 |
| | 3700994·4 | 370125774 |
| | 263·34 | 24671162600 |
| | 3701257·74 | 22210074604 |
| | 3701521·08 | 246108809600 |
| | 158·004 | 222111173304 |
| | 3701679·084 | 23997636296 |
| | 3701837·088 | 22211221613 |
| | 15·8004 | 1786414683 |
| | 3701852·8884 | 1480748782 |
| | 3701868·6888 | 305665901 |
| | 1·5800 | 296149766 |
| | 3701870·268\|8 | 9516135 |
| | 3701871·849 | 7403744 |
| | 105 | 2112391 |
| | 3701871·95\|4 | 1850936 |
| | 3701872·06 | 261455 |
| | 2 | 259131 |
| | 3701872·0\|8 | 2324 |
| | 3\|7\|0\|18\|7\|2\|·1 | 2221 |
| | | 103 |
| | | 74 |
| | | 29 |
| | | 26 |
| | | 3 |

Therefore

$$x = \cdot 31666482570627$$

It thus appears that as $x$ proceeds from $x = 0$ to $x = 1$, the polynomial $X_2$ reaches zero before $X_1$; so that $X_1$ continues *minus* till after $X_2$ has vanished and become *plus*. Hence immediately before and after the evanescence of $X_2$ the signs are

| $X_1$ | $X_2$ | $X_3$ |
|---|---|---|
| $-$ | $-$ | |
| | | $+$ |
| $-$ | $+$ | |

Consequently in the passage of $x$ from 0 to 1 one variation is lost in the series $X_2$, $X_3$, $X_4$, and no more than one, because $X_2$ vanishes but once in the interval (73). But in the passage over the same interval a variation is lost also in the preceding part of the series, that is in $X$, $X_1$, $X_2$. Hence there is a loss of two variations in the entire series, and consequently there are two real roots in the interval $[0, 1]$.

(124.) These roots are remarkably close together. They separate from one another and become distinct only at the sixth decimal place. This we learn from a principle that will be established in the treatise on the *Theory and Solution of Equations of the higher orders;* and which may be generally enunciated as follows:

If two roots of any equation are nearly equal, or agree in any number of their leading figures, then each of the derived polynomials when equated to zero will have a root lying between them; and these roots will all agree to a number of figures so great that if that number be diminished by 1, half the remaining number will express the extent of figures to which the nearly equal roots of the proposed equation agree; and they *may* agree to one figure more. *

---

* The author has to state here that Mr. Rutherford, of the Royal Military Academy, Woolwich, lately communicated to him the enunciation of an analogous property; one in fact closely allied to the principle affirmed above. The property is this, viz., that if two roots of an equation are

(125.) In virtue of this principle we infer that as the root of $X_1 = 0$ agrees with the root of $X_2 = 0$ to 12 places of figures, the number of figures in which the sought roots agree must be $\frac{12-1}{2}$, that is *five* at least; which figures are therefore ·31666. Also as the intermediate roots nearly agree in the *thirteenth* figure we may conclude with nearly absolute certainty that the roots which comprehend them have at most but a unit difference in the *sixth* place; and are therefore

$$x = \cdot 316664 \ldots .$$

$$x = \cdot 316665 \ldots .$$

Let us now seek the places of the negative roots of the proposed equation, which roots we know to be real from these obvious considerations, viz., If the sum of the two roots just discovered be added to 312 the result will be the sum of the two remaining roots; and if 2360 be divided by the product of the same the result will be the product of the remaining roots. The divisor here adverted to is clearly but little different from ·1, so that the depressed quadratic will be very nearly

$$x^2 + 312x + 23600 = 0$$

in which $312^2$ is much greater than $4 \times 23600$.

Hence the roots are real.

The negative root of the quadratic $X_2 = 0$, found by adding ·316 . . . to $-\frac{3685194}{26334}$, is evidently between 130 and 140.

---

nearly equal, and if the last of Sturm's remainders be computed, this last remainder will be preceded by twice as many zeros plus one, as there are figures common to both roots. In other words the two numbers, from the addition of which the final remainder arises, will agree to twice as many leading figures as the roots agree in, and one more: and the numbers themselves will always have contrary signs. An exemplification of this is seen at page 113. See also Note (C) at the end.

| | $X$ | $X_1$ | $X_2$ |
|---|---|---|---|
| $x = 0$ | $+$ | $-$ | $-$ |
| $-100$ | $+$ | $+$ | $-$ |
| $-130$ | $-$ | | |
| $-200$ | $+$ | | |

Hence the negative roots are both between $-100$ and $-200$; one being below $-130$ and the other above it.*

(126.) The equation just analysed is one of peculiar difficulty, because of the exceeding closeness of two of its roots. By the general principle stated at (124) the separation of the roots of $X_3 = 0$ and $X_2 = 0$ is from this circumstance delayed till we reach a figure unusually remote; so that the present example is a very unfavorable one for our method. Nevertheless the whole of the work employed in the foregoing analysis is considerably less than that which would be involved in the calculation of $X_3$ and $X_4$ by the direct method of Sturm, as the student will find if he have the courage to undertake the requisite multiplications.† We have executed all these for the purpose of instituting the comparison, and have exhibited the operations in Note (C) at the end.

(127.) We shall now proceed to the actual development of the roots whose situations have thus been discovered. Those in the interval [0, 1] have already been determined, up to the point at which they separate; but we shall evolve them by the approximative method; carrying on the approximation to two or three figures beyond that at which the separation commences.

---

* The situations of these roots are also easily obtained from the depressed quadratic above.

† This statement would still be true though the developments at pages 184 and 223 were carried on without cutting off the superfluous figures till a much more advanced stage. In example 3 the analysis by the method there adopted is very rapidly effected, while the labour of computing the last two remainders of Sturm which that method supersedes would be very little short of that required for the like computation in the present example, on account of the magnitude of the coefficients.

### *Development of the Two Roots in the interval* [0, 1].

<table>
<tr><td>1</td><td>312</td><td>23337</td><td>— 14874</td><td>— 2360·0000 (·3166644731</td></tr>
<tr><td></td><td>·3</td><td>93·69</td><td>7029·207</td><td>— 2353·4379</td></tr>
<tr><td></td><td>———</td><td>———</td><td>———</td><td>——— 1</td></tr>
<tr><td></td><td>312·3</td><td>23430·69</td><td>— 7844·793</td><td>— 6·56210000</td></tr>
<tr><td></td><td>·3</td><td>93·78</td><td>7057·341</td><td>— 5·51237279</td></tr>
<tr><td></td><td>———</td><td>———</td><td>——— 1</td><td>——— 2</td></tr>
<tr><td></td><td>312·6</td><td>23524·47</td><td>— 787·452</td><td>— 1·049727210000</td></tr>
<tr><td></td><td>·3</td><td>93·87</td><td>236·214721</td><td>— 1·039281237264</td></tr>
<tr><td></td><td>———</td><td>——— 1</td><td>———</td><td>——— 3</td></tr>
<tr><td></td><td>312·9</td><td>23618·34</td><td>— 551·237279</td><td>— 10445972736</td></tr>
<tr><td></td><td>·3</td><td>3·1321</td><td>236·246043</td><td>— 10346656895</td></tr>
<tr><td></td><td>——— 1</td><td>———</td><td>——— 2</td><td>——— 4</td></tr>
<tr><td></td><td>313·2</td><td>23621·4721</td><td>— 314·991236</td><td>— 99315841</td></tr>
<tr><td></td><td>1</td><td>3·1322</td><td>141·777696456</td><td>— 98768403</td></tr>
<tr><td></td><td>———</td><td>———</td><td>———</td><td>——— 5</td></tr>
<tr><td></td><td>313·21</td><td>23624·6043</td><td>— 173·213539544</td><td>— 547438</td></tr>
<tr><td></td><td>1</td><td>3·1323</td><td>141·788973528</td><td>— 534262</td></tr>
<tr><td></td><td>———</td><td>——— 2</td><td>——— 3</td><td>——— 6</td></tr>
<tr><td></td><td>313·22</td><td>23627·7366</td><td>— 31·424566016</td><td>— 13176</td></tr>
<tr><td></td><td>1</td><td>1·879476</td><td>14·180137857</td><td>— 11830</td></tr>
<tr><td></td><td>———</td><td>———</td><td>———</td><td>——— 7</td></tr>
<tr><td></td><td>313·23</td><td>23629·616076</td><td>— 17·24442815|9</td><td>— 1346</td></tr>
<tr><td></td><td>1</td><td>1·879512</td><td>14·18025064</td><td>— 1294</td></tr>
<tr><td></td><td>——— 2</td><td>———</td><td>——— 4</td><td>——— 8</td></tr>
<tr><td></td><td>313·24</td><td>23631·495588</td><td>— 3·06417752</td><td>— 52</td></tr>
<tr><td></td><td>6</td><td>1·879548</td><td>1·41803747</td><td>— 50</td></tr>
<tr><td></td><td>———</td><td>——— 3</td><td>———</td><td>—</td></tr>
<tr><td></td><td>313·246</td><td>23633·375136</td><td>— 1·6461400|5</td><td>— 2</td></tr>
<tr><td></td><td>6</td><td>·187959</td><td>1·4180386</td><td>— 2</td></tr>
<tr><td></td><td>———</td><td>———</td><td>——— 5</td><td>—</td></tr>
<tr><td></td><td>313·252</td><td>23633·56309|5</td><td>— ·2281015</td><td></td></tr>
<tr><td></td><td>6</td><td>·18796</td><td>945360</td><td></td></tr>
<tr><td></td><td>———</td><td>———</td><td>———</td><td></td></tr>
<tr><td></td><td>313·258</td><td>23633·7510|6</td><td>— ·133565|5</td><td></td></tr>
<tr><td></td><td>6</td><td>·1879</td><td>94536</td><td></td></tr>
<tr><td></td><td>——— 3</td><td>——— 4</td><td>——— 6</td><td></td></tr>
<tr><td></td><td>|3|1|3|·2|6|4</td><td>23633·9390</td><td>— 39030</td><td></td></tr>
<tr><td></td><td></td><td>188</td><td>9454</td><td></td></tr>
<tr><td></td><td></td><td>———</td><td>———</td><td></td></tr>
<tr><td></td><td></td><td>23633·957|8</td><td>— 2957|6</td><td></td></tr>
<tr><td></td><td></td><td>19</td><td>945</td><td></td></tr>
<tr><td></td><td></td><td>———</td><td>——— 7</td><td></td></tr>
<tr><td></td><td></td><td>23633·97|7</td><td>— 2013</td><td></td></tr>
<tr><td></td><td></td><td>2</td><td>165</td><td></td></tr>
<tr><td></td><td></td><td>——— 5</td><td>———</td><td></td></tr>
<tr><td></td><td></td><td>2|3|6|3|4|·0|0</td><td>— 184|8</td><td></td></tr>
<tr><td></td><td></td><td></td><td>17</td><td></td></tr>
<tr><td></td><td></td><td></td><td>——— 8</td><td></td></tr>
<tr><td></td><td></td><td></td><td>— 1|6|8</td><td></td></tr>
</table>

If in the sixth decimal place of the root we had put 5 instead of 4, the absolute number would have changed sign, showing that the two roots separate at that place.[*] Putting then 5 for 4, the remaining development of the second root would be as follows:

```
2|3|6|3|4|0        — 2281015          — 547438 (51784
                     1181700          — 549658
                     ———————            ————— 6 †
                   — 109931|5             2220
                     118170               1060
                     ———— 6               ——— 7
                       8238               1160
                       2363               1023
                       ————               ——— 8
                     1060|1                137
                       236                 131
                       ——— 7               ———
                     1296                    6
                      165                    6
                      ———                    —
                     146|1
                      16
                      —— 8
                     162
                       2
                      ——
                     1|6|4
```

Consequently the two roots are

$$x = \cdot 3166644731$$

$$x = \cdot 3166651784$$

---

[*] Since by the foregoing analysis, and the principle announced at page 224, we know that the roots cannot separate till the 4 is reached, we need not commence a search after this separation till our arrival at that figure.

[†] We see that, as stated above, the absolute number in this transformation has changed sign; showing that 5 has a single root between ·000004 and ·000005.

*Development of the Root* [— 100,—130]

We shall change this negative into a positive root, by changing the alternate signs of the proposed equation

```
1     — 312      23337          14874          — 2360  (126·3166644731
        100    — 21200         213700        22857400
       ————      —————         ——————        ———————— 1
      — 212       2137         228574     — 22859760
        100    — 11200       — 906300     — 16795720
       ————      —————       ———————— 1   ———————— 2
      — 112     — 9063       — 677726      — 6064040
        100     — 1200       — 102060      — 5759280
       ————      ————— 1     ————————     ——————————— 3
      —  12    — 10263       — 839786    — 3047600000
        100       2160       — 110860    — 2887258779
       ———— 1    —————       ———————— 2   ——————————— 4
         88     — 8103       — 950646     — 160341221
         20       2560         — 9234      — 96217953
       ————      —————       ————————      ————————— 5
        108     — 5543       — 959880      — 64123268
         20       2960         — 2754      — 57729970
       ————      ————— 2     ———————— 3    ————————— 6
        128     — 2583       — 962634       — 6393298
         20       1044        214·407       — 5772964
       ————      —————       ————————       —————————
        148     — 1539    — 962419·593       — 620334
         20       1080        231·741        — 577296
       ———— 2    —————     ——————————— 4      ———————
        168      — 459    — 962187·852        — 43038
          6       1116          8·323         — 38486
       ————      ————— 3   ———————————        ———————
        174        657   — 962179·52|9         — 4552
          6      57·69           8·34          — 3848
       ————     ——————     ——————————— 5       ——————
        180     714·69    — 962171·19           — 704
          6      57·78           5·02           — 673
       ————     ——————     ———————————          —————
        186     772·47    — 962166·1|7           — 31
          6      57·87            5·0            — 29
       ———— 3   —————— 4   ——————————— 6         ————
        192     830·34     — 962161·2             — 2
         ·3       1·93             ·5             — 1
       ————     ——————     ———————————           ————
      192·3    832·2|7   —9|6|2|1|6|0|·7          — 1
         ·3         19
       ————     ——————
      192·6     834|·2
         ·3          2
       ————     —————— 5
      192·9     |8|3|6
         ·3
      ————— 4
    1|9|3|·2
```

Hence the root is

$$x = 126·3166644731$$

*Development of the root* [—130, —200].

As before, we shall convert this into a positive root; and to avoid repeating the first transformation, we shall commence with the results marked 1 in the preceding development; these being the coefficients of the equation whose roots are those of the proposed diminished by 100.

| | | | |
|---|---|---|---|
| 88 | —10263 | —677726 | —22859760 (86·3166651784 |
| 80 | 13440 | 254160 | —33885280 |
| ——— | ——— | ——— | ——— 2* |
| 168 | 3177 | —423566 | 11025520 |
| 80 | 19840 | 1841360 | 10369440 |
| ——— | ——— | ——— 2 | ——— 3 |
| 248 | 23017 | 1417794 | 6560800000 |
| 80 | 26240 | 310446 | 6212670021 |
| ——— | ——— 2 | ——— | ——— 4 |
| 328 | 49257 | 1728240 | 348129979 |
| 80 | 2484 | 325566 | 208858506 |
| ——— 2 | ——— | ——— 3 | ——— 5 |
| 408 | 51741 | 2053806 | 139271473 |
| 6 | 2520 | 17084·007 | 125370032 |
| ——— | ——— | ——— | ——— 6 |
| 414 | 54261 | 2070890·007 | 13901441 |
| 6 | 2556 | 17122·941 | 12539269 |
| ——— | ——— 3 | ——— 4 | ——— 7 |
| 420 | 56817 | 2088012·948 | 1362172 |
| 6 | 129·69 | 572·107 | 1253949 |
| ——— | ——— | ——— | ——— 8 |
| 426 | 56946·69 | 2088585·05\|5 | 108223 |
| 6 | 129·78 | 572·15 | 104496 |
| ——— 3 | ——— | ——— 5 | ——— |
| 432 | 57076·47 | 2089157·21 | 3727 |
| ·3 | 129·87 | 343·33 | 2090 |
| ——— | ——— 4 | ——— | ——— |
| 432·3 | 57206·34 | 2089500·5\|4 | 1637 |
| ·3 | 4·33 | 343·3 | 1462 |
| ——— | ——— | ——— 6 | ——— |
| 432·6 | 57210·6\|7 | 2089843\|·8 | 175 |
| ·3 | 4·3 | 34·3 | 166 |
| ——— | ——— | ——— | ——— |
| 432·9 | 57215\|·0 | 2089878\|1 | 9 |
| ·3 | 4 | 34 | 8 |
| ——— 4 | ——— 5 | ——— | ——— |
| 433·2 | 57219 | 2089912 | 1 |
| 1 | 2 | 3 | |
| ——— | ——— | ——— 7 | |
| 4\|3\|3\|·2 | \|5\|7\|2\|2\|1 | 2\|0,8\|9\|9\|1\|5 | |

---

* The change of sign in the absolute number of this transformation

Consequently the root is $x = 186{\cdot}3166651784$.

It thus appears that the three roots of the proposed equation are as follow:

$$\begin{array}{rl} \cdot3166644731 & \\ \cdot3166651784 & \\ -126{\cdot}3166644731 & \\ -186{\cdot}3166651784 & \\ \hline -312 \qquad\qquad\qquad & = \text{Sum of the roots} \end{array}$$

(128.) As before remarked, the example now so fully discussed is one of considerable difficulty. It has been framed expressly for the purpose of putting the modern methods and resources to a severe test. The actual occurrence of such an equation, in any mathematical enquiry, is extremely improbable; but it furnishes an excellent exemplification of the entire efficiency of the combined theories of Sturm and Horner, even in those extreme and critical cases which are so admirably calculated to detect the weak points of a defective principle *.

Beyond these two theories—the preparatory analysis of Sturm, and the subsequent continuous development of Horner—nothing is absolutely indispensable to complete the general problem of the solution of numerical equations of all degrees. Yet there is room for the advantageous introduction of other considerations, and other principles, into the discussion of this important problem, in combination with the methods of Sturm and Horner; and which enable us to bring certain ambiguities, which occasionally present themselves in our transformed equations, to a more ready test than an appeal to Sturm's remainders would furnish.

---

shows that a single root of $\mathfrak{X}$ lies between 20 and 80. The substitution of 9 for 8 would again change this sign from plus to minus, showing that another root lies between 80 and 90; that is, that 8 is the true first figure of a root of $\mathfrak{X}$.

* The equation referred to was sent to the author by Mr. Lockhart, whose laborious researches into the doctrine of numerical equations have led to many interesting and valuable results.

The more advanced student will foresee that we are adverting, in these observations, to the enquiries of Budan and Fourier; of which an ample examination and analysis will be found in the volume on Equations of the Higher Degrees which is supplementary to the present treatise. The ambiguities to which we allude arise from the entrance of imaginary roots into our equations; and from the circumstance that, in proceeding from one transformation to another, the changes which the signs undergo, and from which changes the character of the roots,—when known to be real—may always be inferred (30), are the same for a pair of imaginary roots as for a pair of real roots. In equations of a high degree, where pairs of imaginary roots and pairs of contiguous real roots may enter, we might, in the course of development, confound an imaginary with a real interval. Sturm's theorem, applied to the doubtful equation, would always remove the difficulty.

But, as we have already seen, this theorem, when the equation analysed by it has large coefficients, and is, moreover, of a high degree, is laborious; and should therefore be appealed to only in cases of absolute necessity. The researches alluded to supply us with some useful helps in these circumstances.

(129.) The labour of computing Sturm's remainders, however, falls principally upon the last two of the series. These we have dispensed with in the present chapter; and have shown how to obtain the information supplied by them with less difficulty, whenever the coefficients of the equation are so large as to render this labour especially irksome. In estimating the comparative advantages of the two modes of procedure, it must be borne in mind that the work of deducing the remainders increases with the magnitude of the coefficients, while the labour involved in the other method increases only with the extent to which neighbouring roots agree in their leading figures; so that in the majority of the cases where the coefficients would greatly complicate Sturm's final remainders, but little is added to the trouble of separating the roots by the process here employed, in which process, it should be remarked, all the work of the analysis is actually exhibited, in continuous operations easily revised.

In the direct method of Sturm for analysing an equation, attention is confined exclusively to the *horizontal* rows of signs furnished by the series of remainders. In the modification here proposed, the *vertical* rows of signs are also submitted to examination, and are found to supply useful particulars respecting the constitution of the equation.

As to the form given in this work to the operations for deducing Sturm's remainders, we may remark that it has been adopted because the simple and uniform character of the several steps of the calculation seemed to recommend it as better adapted to the learner, and more easily borne in the memory, than any other we could devise. We have elsewhere* given a more compact form to the computation; and one which, in the higher equations, where the remainders are more numerous and the work longer, is on account of its greater brevity of more convenient application. Nearly the same amount of actual numerical labour enters into both methods; but in long calculations economy of space merely is by no means matter of indifference. In such calculations the more brief and compact the type of the operations is, the better; provided that type distinctly present the leading features of the numerical work, and the manner in which the successive steps of the process are connected together; thus furnishing an efficient working model for every case of actual practice.

(130.) We shall conclude these researches with a few practical examples for the further exercise of the student in the analysis and solution of biquadratic equations; and shall then terminate the work by briefly showing, as was promised at the outset (6), that every biquadratic equation has necessarily a root, either real or imaginary.

### EXAMPLES FOR EXERCISE.

1. One root of the equation

$$x^4-12x^2+12x-3=0$$

---

* See the author's *Mathematical Dissertations*, Diss. IV.

lies between 2 and 3. Required the development of it to six places of decimals.

Ans. $x = 2{\cdot}858033$.

2. One root of the equation

$$x^4 + 4x^3 - 4x^2 - 11x + 4 = 0$$

lies between 1 and 2. Required the development of it to eight places of decimals.

Ans. $x = 1{\cdot}63691356$.

3. Required the complete analysis and solution of the equation

$$x^4 - 12x^2 + 12x - 3 = 0$$

The roots, to eight places of decimals, are as follow:

8/1 $2{\cdot}8580\cancel{3}308, \quad {\cdot}44327694, \quad {\cdot}60601831, \quad -3{\cdot}90737855$

4. Required the analysis and solution of the equation

$$25x^4 - 298x^3 + 576x^2 + 281x - 26 = 0$$

The roots are as follow:

$1{\cdot}283137, \quad 9{\cdot}65553, \quad 1{\cdot}06049, \quad -{\cdot}079154$

---

### *Simpson's Solution of a Biquadratic Equation.*

(131.) Let the biquadratic equation be

$$x^4 + 2ax^3 + bx^2 + cx + d = 0$$

Assume the first member of it equal to

$$(x^2 + ax + \mathrm{A})^2 - (\mathrm{B}x + \mathrm{C})^2$$

where A, B, C are unknown quantities, which, by the method of indeterminate coefficients (*Algebra*, art. 175), are related to the coefficients of the proposed equation by the three following conditions:

$$2A + a^2 - B^2 = b \therefore B^2 = 2A + a^2 - b \quad . \; . \; . \quad [1]$$

$$2aA - 2BC = c \therefore 2BC = 2aA - c \quad . \; . \; . \; . \; . \quad [2]$$

$$A^2 - C^2 = d \therefore C^2 = A^2 - d \quad . \; . \; . \; . \; . \; . \quad [3]$$

If the first members of [1] and [3] be multiplied together, four times the product will be equal to the square of the first member of [2]. Hence, employing the second members instead, we have

$$8A^3 + 4a^2A^2 - 8Ad - 4a^2d - 4bA^2 + 4bd = 4a^2A^2 - 4acA + c^2$$

$$\therefore 8A^3 - 4bA^2 + 4(ac - 2d)A = 4a^2d - 4bd + c^2$$

A may be found from this cubic equation; and thence

$$B = \sqrt{2A + a^2 - b}$$

$$\text{and } C = \frac{2aA - c}{2B} = \frac{2aA - c}{2\sqrt{2A + a^2 - b}}$$

Now by the original assumption we have

$$x^2 + ax + A = Bx + C$$

or

$$x^2 + (a - B)x + (A - C) = 0$$

from which quadratic $x$ may be determined in quantities already known. Hence every biquadratic equation has a root (see p. 5).

### *Scholium.*

For the means of determining the remaining roots of a biquadratic equation, when two are known, the student may consult Note (B) at the end.

## NOTE (A), page 206.

### (132.) *New Method of discovering the remaining Roots of a Cubic Equation when one root is determined.*

Let
$$x^3 + px + q = 0 \quad \ldots . [1]$$

be any cubic equation, when deprived of its leading coefficient and of its second term. Then by the formula of Cardan, (*Algebra*, p. 159,) if we put

$$\left.\begin{aligned} y &= \left\{-\frac{q}{2} + \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right\}^{\frac{1}{3}} \\ z &= \left\{-\frac{q}{2} - \sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)}\right\}^{\frac{1}{3}} \end{aligned}\right\} \quad \ldots [2]$$

one root of the equation will be

$$x = y + z$$

And it is shown in the treatise on the *Theory of Equations*, that the other roots will be

$$-\frac{1}{2}\left\{x \pm \sqrt{-3}\,(y - z)\right\} \quad \ldots . [3]$$

Now from [2] it is plain that the following conditions have place, viz.

$$\left.\begin{aligned} y^3 - z^3 &= 2\sqrt{\left(\frac{1}{4}q^2 + \frac{1}{27}p^3\right)} \\ y^3 + z^3 &= -q, \; yz = -\frac{1}{3}p \end{aligned}\right\} \quad \ldots [4]$$

And since

$$\frac{y^3 - z^3}{y - z} = y^2 + yz + z^2$$

and

$$\frac{y^3 + z^3}{y + z} = y^2 - yz + z^2$$

it follows that

$$\frac{y^3 - z^3}{y - z} = \frac{y^3 + z^3}{y + z} + 2yz$$

that is, by the conditions [4]

$$\frac{2\sqrt{\left(\frac{1}{4}q^2+\frac{1}{27}p^3\right)}}{y-z}=-\frac{q}{x}-\frac{2}{3}p$$

Consequently

$$y-z=-\frac{2\sqrt{\left(\frac{1}{4}q^2+\frac{1}{27}p^3\right)}}{\frac{q}{x}+\frac{2}{3}p}$$

$$=-\frac{3x\sqrt{\left(q^2+\frac{4}{27}p^3\right)}}{3q+2px}$$

Therefore [3] the other two roots of the proposed cubic are

$$\pm\frac{1}{2}\cdot\frac{3x\sqrt{\left(-3q^2-\frac{4}{9}p^3\right)}}{3q+2px}-\frac{1}{2}x$$

$$=\pm\frac{x\sqrt{\left(-\frac{27}{4}q^2-p^3\right)}}{3q+2px}-\frac{1}{2}x$$

$$=\pm\frac{x\sqrt{(-6{\cdot}75q^2-p^3)}}{3q+2px}-\frac{1}{2}x\;\ldots\ldots\;[5]$$

When $-27q^2-4p^3$, or $-6{\cdot}75q^2-p^3$ happens to be a complete square, these remaining roots are very easily computed by this formula; and even when such is not the case, and we have therefore to perform the extraction of the square root, the formula will generally furnish the two remaining roots, after one is found, more expeditiously than by actual development. An interesting inference, too, is immediately supplied by this formula, viz., that when all the roots of a cubic equation of the form [1] are commensurable, the expression

$$-(27q^2+4p^3)\text{ or }-(6{\cdot}75q^2+p^3)$$

must be a perfect square. The converse of this, however, has not place.

(133.) We shall now apply this formula to the equation

$$x^3 - 7x + 7 = 0$$

which is discussed at length by Lagrange;[*] and was formerly considered as an equation of some difficulty. The roots of it have been found, at page 189, to be

$$1\cdot356895,\ 1\cdot692021,\ -3\cdot048916$$

In this example

$$-6\cdot75q^2 = -6\cdot75 \times 49 = 330\cdot75\,; \quad -p^3 = 343$$

$$\therefore -6\cdot75q^2 - p^3 = 12\cdot25 = 3\cdot5^2$$

Consequently, by the formula [*5*],

$$\pm\frac{3\cdot5x}{21-14x} - \frac{1}{2}x, \text{ or rather } \pm\frac{x}{6-4x} - \frac{1}{2}x$$

will furnish both the remaining roots when either of the three, given above, are substituted for $x$. Let us take the first root, remembering, however, that as the final figure *5*, does not complete that root, the figure in the quotient which immediately depends upon this *5* will be liable to error.

```
        6
  4x = 5·42758
       ───────
        57242) 1·356895 (2·37045
               1·14484
               ───────
                212055
                171726
                ──────
                 40329
                 40069
                 ─────
                   260
                   229
                   ───
                    31
                    29
                    ──
                     2
```

The last figure *5*, in the above quotient, cannot be depended on as strictly accurate, for reasons we have already adverted to. By first adding it to, and then subtracting it from, $-\frac{1}{2}x$ or $-\cdot678447$, we have

```
                 —·678447
                  2·37045
                  ───────
       Sum =      1·69200 = second root
                  ───────
Difference =    —3·04890 = third root
```

---

[*] Résolution des Equations Numériques, 1826, pp. 33, 117.

(134.) When $-(6{\cdot}75q^2 + p^3)$ is not a perfect square, a formula, somewhat more convenient for the calculation of the remaining roots, may be employed. It is deduced as follows:

Since (page 236),

$$-\frac{q}{x} = y^2 - yz + z^2, \text{ and } -\frac{1}{3}p = yz$$

we have

$$(y - z)^2 = -\frac{q}{x} + \frac{1}{3}p.$$

Hence, by [3], the remaining roots are

$$\pm\frac{1}{2}\sqrt{\left(\frac{3q}{x} - p\right)} - \frac{1}{2}x \quad \ldots\ldots \quad [6]$$

We would recommend the student to apply the formula [6] to the discovery of the second and third roots of the equation 7, at page 185, after having deprived it of its leading coefficient and second term, by the usual transformation.

From a remark made by Mr. Lockhart, at page 18 of his *Resolution of Equations*, 1837, it would seem that that gentleman is in possession also of a method for finding the remaining roots of a cubic equation when one is known, in the case in which $-6{\cdot}75q^2 - p^3$ is a complete square. The remark alluded to is as follows:

" It is an interesting circumstance respecting the equation

$$x^3 - bx = c$$

that if the quantity $b^3 - 6{\cdot}75c^2$ is a square, the roots of the equation may all be expressed by either of two fractions wherein there is no mention of a surd.

" Thus let $x^3 - 19x = 30$. One fraction is

$$-\frac{163x + 361}{57x + 107}$$

the roots are 5, $-3$, $-2$, either of which substituted for $x$ gives one of the others."

We are not informed by the venerable author how this fraction is obtained. But it is clearly not implied in the general formula investigated above, which gives a different, and indeed a simpler expression for the two roots. It is this:

$$\pm\frac{14x}{45 + 19x} - \frac{1}{2}x$$

It is pretty obvious that the quantity with which $-\frac{1}{3}x$ is connected, whether in formula [5] or in formula [6], is half the difference of the remaining roots. For $-\frac{1}{3}x$ is the half sum of those roots; and the quantity referred to is that which, when added to and subtracted from this half sum, gives the roots themselves. In fact these formulas merely exhibit the solution of the quadratic equation involving the roots sought.

By equating the two expressions [5] and [6], and squaring, we have

$$-27q^2-4p^3=\left(\frac{3q}{x}+2p\right)^2\left(\frac{3q}{x}-p\right)$$

Now the first member of this equation expresses the product of the squares of the differences of the three roots, as is shown in the *Theory of Equations of the Higher Orders.* But when two of the roots are equal, this product must be zero; hence in this case the second member of the foregoing equation must be zero; and consequently

$$\left(\frac{3q}{x}+2p\right)^2=0$$

or

$$\frac{3q}{x}-p=0$$

From the first of these

$$x=-\frac{3q}{2p},\ x=-\frac{3q}{2p}$$

From the second $x=\frac{3q}{p}.$

The former are therefore the two equal roots, and this latter is the third root.

The equal roots are usually expressed under a surd form, as at page 70. In the present case the equal roots are each $+\sqrt{\frac{p}{3}}$, or each $-\sqrt{\frac{p}{3}}$, according as $q$ is positive or negative; so that when equal roots exist in the equation [1] $\frac{p}{3}$ is necessarily a square, viz., the square of one of those roots. This might have been inferred from (75).

## NOTE (B), page 235.

### (135.) *When two roots of a Biquadratic Equation are determined to discover the remaining Roots.*

Let the biquadratic equation, when deprived of its leading coefficient and of its second term, be

$$x^4 + px^2 + qx + r = 0 \ . \ . \ . \ [1]$$

and let two of its roots be $x_1$, $x_2$, the two remaining roots being $x_3$, $x_4$. As the sum of these must be equal to the sum of the former pair with changed signs, it follows that the quadratic equation, involving the roots $x_3$, $x_4$, will be

$$x^2 + (x_1 + x_2)\,x + x_3x_4 = 0.$$

Hence the roots themselves will be

$$-\frac{x_1 + x_2}{2} \pm \sqrt{\left\{\left(\frac{x_1 + x_2}{2}\right)^2 - x_3x_4\right\}}$$

$$= -\frac{x_1 + x_2}{2} \pm \sqrt{\left\{\left(\frac{x_1 - x_2}{2}\right)^2 + x_1x_2 - x_3x_4\right\}}$$

Now as the coefficient $q$ is the sum of the products of the roots taken three and three, with changed signs, we have

$$-(x_1 + x_2)\,x_3x_4 - (x_3 + x_4)\,x_1x_2 = q$$

but $(x_3 + x_4) = -(x_1 + x_2)$. Consequently

$$(x_1 + x_2)\,(x_1x_2 - x_3x_4) = q$$

$$\therefore x_1x_2 - x_3x_4 = \frac{q}{x_1 + x_2}$$

Hence, by substitution in the foregoing expression, we have for the two roots $x_3, x_4$, the following formula, viz.:

$$-\frac{x_1 + x_2}{2} \pm \sqrt{\left\{\left(\frac{x_1 - x_2}{2}\right)^2 + \frac{q}{x_1 + x_2}\right\}} \ . \ . \ . \ . \ [2]$$

As an example of the application of this formula, let the equation

$$x^4 - 55x^2 - 30x + 504 = 0$$

be proposed, two roots of which are 3 and 7.

The formula gives for the remaining roots

$$-5 \pm \sqrt{4 + \frac{-30}{10}} = -5 \pm 1 = -4 \text{ and} -6$$

Again; two roots of the equation

$$x^4 - 3x^2 + 6x - 2 = 0$$

are $-1 + \sqrt{2}$, and $-1 - \sqrt{2}$. For the other roots we have, by the formula,

$$1 \pm \sqrt{2 + \frac{6}{-2}} = 1 \pm \sqrt{-1}$$

This formula will in general furnish the remaining roots o a biquadratic equation, when two are ascertained, more expeditiously than they can be determined by actual development. It will be found more especially valuable in those difficult cases where, as in example 4, page 221, there are indications of two roots lying very closely together. If the other two roots, those whose places are distinctly marked, be developed, and then the foregoing formula employed, the nearly equal roots, if such exist, will be determined with comparatively little trouble. It is obvious that the operation necessary to reduce any biquadratic to the form [1] may be dispensed with. The two roots may be developed as in the text, and then each of them increased by the fourth part of the second coefficient divided by the first. The results, substituted for $x_1$ and $x_2$ in the formula [2], will give the other roots increased in like manner.

It may further be remarked that in performing the operations within the brackets, in the formula [2], we need not compute to a greater extent of decimals than the roots $x_1$ and $x_2$ have been carried to; nor in the subsequent extraction of the square root should this limit be exceeded. And, as in the case of cubics, the already computed roots should be carried a decimal or two beyond the place up to which accuracy in the remaining roots is expected.

When the two roots $x_1, x_2$ are equal, we know from (75) that they must be also commensurable; and then the calculation of the remaining roots becomes very simple, our formula [2] being in that case

$$-x_1 \pm \sqrt{\frac{q}{2x_1}}.$$

## NOTE (C.)

(136.) *Computation of the Remainders* $X_3$ *and* $X_4$ *in the Example at page* 221 *by the Method of Sturm.*

As the coefficients of $X_2$ computed at page 221 admit of division by 2, we shall employ them reduced by this divisor.

$$\begin{array}{r|rrr}
13167 & +1842597 & -584806 & \\
2 & -468 & -23337 & +7437 \\
\hline
 & 3685194 & -1169612 & \\
 & -6162156 & -307278279 & +97922979 \\
\hline
\end{array}$$

$$\begin{array}{r|rr}
-2476962 & -308447891 & +97922979 \\
13167 & -1842597 & +584806 \\
\hline
 & -4061333380797 & +1289351864493 \\
 & 4564042750314 & -1448542239372 \\
\hline
 & 502709369517 & -159190374879
\end{array}$$

$$\therefore X_3 = 502709369517x - 159190374879$$

$$\begin{array}{r|rr}
502709369517 & -159190374879 & \\
13167 & -1842597 & +584806 \\
\hline
 & -2096059666031793 & \\
 & -926290776143915649 & +293987455549758702 \\
\hline
\end{array}$$

$$\begin{array}{r|r}
-928386835809947442 & +293987455549758702 \\
502709369517 & +159190374879 \\
\hline
 & 147790248425326259703904286934 \\
 & -147790248425314154889267109518 \\
\hline
 & 12104814637177416
\end{array}$$

$$\therefore X_4 = +$$

The multiplications which furnish the results here given are exhibited in the following pages.

13167
468
105336
79002
52668
6162156

23337
13167
163359
140022
23337
70011
23337
307278279

13167
7437
92169
39501
52668
92169
97922979

* * *

— 308447891
13167
2159135237
1850687346
308447891
925343673
308447891
— 4061333380797

97922979
13167
685460853
587537874
97922979
293768937
97922979
1289351864493

— 2476962
— 1842597
17338734
22292658
12384810
4953924
9907848
19815696
2476962
4564042750314

— 2476962
584806
14861772
19815696
9907848
19815696
12384810
—1448542239372

* * *

— 159190374879
13167
1114332624153
955142249274
159190374879
477571124637
159190374879
—2096059666031793

502709369517
— 1842597
3518965586619
4524384325653
2513546847585
1005418739034
2010837478068
4021674956136
502709369517
— 926290776143915649

502709369517
584806
3016256217102
4021674956136
2010837478068
4021674956136
2513546847585
293987455549758702

* * *

```
      2939874555549758702
             502709369517
      -------------------
     20579121888848310914
     2939874555549758702
   14699372777748793510
  26458870999947828318
 17639247333298552212
 8819623666649276106
26458870999947828318
20579121888848310914
5879749111099517404
14699372777748793510
------------------------------
147790248425326259703904286934
------------------------------
```

```
      — 928386835809947442
              159190374879
        ------------------
       8355481522289526978
      6498707850669632094
     7427094686479579536
    3713547343239789768
   6498707850669632094
  2785160507429842326
 8355481522289526978
 928386835809947442
8355481522289526978
4641934179049737210
928386835809947442
----------------------------------
— 147790248425314154889267109518
----------------------------------
```

(137.) The risk of error attendant upon these long multiplications is very considerable—much more than that to which the developments at pages 184 and 223 are exposed. There is moreover a tedious monotony in such operations which renders them far more repulsive than the more diversified processes referred to. On the score of actual labour it will now be tolerably apparent that the method in the text is to be preferred even in those extreme and unusual cases which, like that here considered, render the comparison of the two processes particularly favorable to the direct method of Sturm.*

The example at page 214, if analysed by this direct method, will be found to involve the computer in multiplications but little inferior in extent to those above; for the magnitude of these always depends upon that of the original coefficients, whereas the length of the developments by which the calculation of $X_3$ and $X_4$ is dispensed with is only in proportion to the proximity of the roots.

In the *Mathematical Dissertations* we have suggested a contrivance for greatly reducing these long multiplications; and we have here described a mode of procedure by which the more tedious of these may be avoided altogether. It is only by abridging this department of the work that the method of Sturm, so simple and beautiful in theory, can arrive at perfection as a practical formula, for equations of a high degree and with large coefficients.

The method adopted in the present volume was briefly, and indeed but imperfectly, hinted at in Note II. of the work just referred to; in articles 136 and 137 of which are some remarks which the readers of this treatise will see occasion to modify.

The example to which the present NOTE refers proves the possibility of equations, whose coefficients are not remarkably large, having roots with many leading figures in common; and the principle stated at (124) shows how we may, from such equations, frame others, a pair of whose roots shall keep together for twice that number of figures. Thus if any two of the equations $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, be combined together, the resulting equation will have a pair of roots which do not separate and become distinct till the *thirteenth* figure is reached.

---

* It might seem that, in justice to the direct method, we ought to have exhibited the developments at pp. 184, 223 without any curtailment of the several columns. But the tediousness and liability to error of the *additions* which give the several products above, and the trouble of performing which cannot actually be made apparent, fully compensate, we think, for the reductions introduced into the developments referred to. But, if these reductions had been postponed sufficiently far to secure accuracy in the root to double the number of figures already computed, still there could be little hesitation in pronouncing the method in the text to be both shorter and simpler, and much less exposed to error, than the process exhibited above.

It may be further remarked here, in reference to that principle, that among those roots of the several derived equations, which thus approximate to one another, some two or more may be found to approach much nearer than is stated in the text. But they *all* agree only to the extent there affirmed; nor can the roots of two *consecutive* remainders agree to a greater extent. It is this that fully justifies the inference at (125).

The proof of the principle is founded upon the connexion between Sturm's final remainder, and the product of the squares of the differences of the roots, which connexion is fully explained in the volume on *Equations of the higher orders*. A somewhat more explicit enunciation of it would be this, viz.—That if the nearly equal roots in two consecutive remainders be subtracted one from the other, the difference will be preceded by twice as many blank places, or zeros; or twice as many, plus one, as the difference of the nearly equal roots of the proposed equation will be preceded by.

THE END.

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